# PUTNAM PRACTICE SET 2

PROF. DRAGOS GHIOCA

*Problem 1.* Let $k \in \mathbb{N}$ and let $a_1, \dots, a_k, b_1, \dots, b_k \in \mathbb{N}$. We know that $\gcd(a_i, b_i) = 1$ for each $i = 1, \dots, k$. We let $M$ be the least common multiple of the numbers $b_1, \dots, b_k$ and also, we let $D$ be the greatest common divisor of the numbers $a_1, \dots, a_k$. Then prove that the greatest common divisor of all the numbers $\frac{a_i \cdot M}{b_i}$ for $i = 1, \dots, k$ is equal to $D$.

*Problem 2.* Let $P \in \mathbb{Z}[x]$ be a polynomial of degree $\deg(P) \geq 1$. We let $n(P)$ be the number of all integers $k$ for which $(P(k))^2 = 1$. Prove that $n(P) - \deg(P) \leq 2$.

*Problem 3.* Let $a_1, \dots, a_5$ be real numbers such that

$$a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 = 1.$$

Prove that $\min_{1 \leq i < j \leq 5} (a_i - a_j)^2 \leq \frac{1}{10}$.

*Problem 4.* Let $n$ be a positive integer. Prove that the number

$$\sum_{k=0}^{n} \binom{2n+1}{2k+1} \cdot 8^k$$

is not divisible by 5.

*Problem 5.* Let $n$ be a positive integer, let $a_1, \dots, a_n$ be positive real numbers, and let $q \in (0, 1)$ be a real number. Prove that there exist $n$ real numbers $b_1, \dots, b_n$ satisfying the following properties:

- $a_k < b_k$ for each $k = 1, \dots, n$;
- $q < \frac{b_{k+1}}{b_k} < \frac{1}{q}$ for $k = 1, \dots, n-1$; and
- $b_1 + \cdots + b_n < \frac{1+q}{1-q} \cdot (a_1 + \cdots + a_n)$.

*Problem 6.* For each $n \in \mathbb{N}$, we let $Q_n$ be a square of side length $\frac{1}{n}$. Prove that in a square of side length $\frac{3}{2}$ we can arrange all the squares $Q_n$ such that for any $m \neq n$, the squares $Q_m$ and $Q_n$ are placed so that there are no interior common points for both $Q_m$ and $Q_n$.