PUTNAM PRACTICE SET 8

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Problem 1. Prove that if $a, b, c \in \mathbb{C}$ and the following relations are satisfied:
- $a + b + c = 0$; and
- $|a| = |b| = |c|$,

then $a^3 = b^3 = c^3$.

Can this result be extended to more than 3 complex numbers?

Problem 2. If the series $\sum_{n=1}^{\infty} a_n$ of real numbers converges, does $\sum_{n=1}^{\infty} a_n^3$ converge?

Problem 3. For what pairs (a, b) of positive real numbers we have that the integral

$$\int_b^{\infty} \left(\sqrt{\sqrt{x+a} - \sqrt{x}} - \sqrt{\sqrt{x} - \sqrt{x-b}} \right) dx$$

converges.

Problem 4. For each $n \in \mathbb{N}$, we let S_n be the set of all pairs $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ with the property that $x^3 - 3xy^2 + y^3 = n$.

(a) For each $n \in \mathbb{N}$, prove that either S_n is the empty set, or it has at least 3 elements.
(b) Prove that S_{2021} is the empty set.