PUTNAM PRACTICE SET 3

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Problem 1. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function satisfying the relation:

$$f(x + y + xy) = f(x) + f(y) + f(xy) \text{ for each } x, y, \in \mathbb{R}.$$

Prove that $f(x + y) = f(x) + f(y)$ for each $x, y \in \mathbb{R}$.

Problem 2. Find all positive real numbers a with the property that the equation $\log_a(x) - x = 0$ has exactly one real solution.

Problem 3.

(a) Find all integers $n > 2$ for which there exists an integer $m \geq n$ such that m divides the least common multiple of $m - 1, m - 2, \cdots, m - n + 1$.

(b) Find all positive integers $n > 2$ for which there exists exactly one integer $m \geq n$ such that m divides the least common multiple of $m - 1, m - 2, \cdots, m - n + 1$.

Problem 4. Find the minimum of

$$\max\{a + b + c, b + c + d, c + d + e, d + e + f, e + f + g\}$$

where the real numbers a, b, c, d, e, f, g vary among all the possible nonnegative solutions to the equation $a + b + c + d + e + f + g = 1$.