PUTNAM PRACTICE SET 1

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Problem 1. Find the maximum and the minimum possible value of the product $x_1 \cdot x_2 \cdots x_n$, where the real numbers x_i satisfy the following properties:

- $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$; and
- $x_i \geq \frac{1}{n}$ for each $i = 1, \ldots, n$.

Problem 2. We let $f : [0, 1) \longrightarrow [0, 1)$ be defined by the properties:

$$\begin{cases} f(x) = \frac{f(2x)}{4} & \text{if} \quad 0 \leq x < \frac{1}{2} \\ f(x) = \frac{3+f(2x-1)}{4} & \text{if} \quad \frac{1}{2} \leq x < 1 \end{cases}.$$

Find $f(x)$ for each $x \in [0, 1)$; you may express your answer in terms of the expansion of x in base 2.

Problem 3. Find all real numbers a for which there exist nonnegative real numbers $x_1, \ldots, x_5$ satisfying the following property:

$$\sum_{k=1}^{5} k^{2i-1} \cdot x_k = a^i \text{ for each } i = 1, 2, 3.$$

Problem 4. Let $m \in \mathbb{N}$ and let $a_1, \ldots, a_m \in \mathbb{N}$. Prove that there exists a positive integer $n < 2^m$ and there exist positive integers $b_1, \ldots, b_n$ satisfying the following properties:

(i) for any two distinct subsets $I, J \subseteq \{1, \ldots, n\}$, we have that $\sum_{k \in I} b_k \neq \sum_{\ell \in J} b_\ell$; and

(ii) for each $i = 1, \ldots, m$, there exists a subset $J_i \subseteq \{1, \ldots, n\}$ such that $a_i = \sum_{k \in J_i} b_k$.