PUTNAM PRACTICE SET 11

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Problem 1. Let $p > 3$ be a prime number. Prove that at least one of the numbers from the following list:

$$\frac{3}{p^2}, \frac{4}{p^2}, \frac{5}{p^2}, \dots, \frac{p-2}{p^2}$$

can be written as a sum $\frac{1}{x} + \frac{1}{y}$ for some positive integers x and y.

Problem 2. If $r > s > 0$ and $a > b > c > 0$, prove that

$$a^r b^s + b^r c^s + c^r a^s \ge a^s b^r + b^s c^r + c^s a^r.$$

Problem 3. Find all $f \in \mathbb{C}[x]$ with the property that for each $x \in \mathbb{C}$, we have $f(x)f(2x^2) = f(2x^3 + x)$.

Problem 4. Let $n \in \mathbb{N}$ and let $S_n = \{1, \dots, n\}$. Assume the set $M \subseteq S_n \times S_n$ satisfies the following properties:

- if $(j, k) \in M$ then $1 \le j < k \le n$; and
- if $(j, k) \in M$ then for each $i \in S_n$, we have that $(k, i) \notin M$.

What is the largest possible cardinality of the set M?