Empirical distribution along geodesics in exponential last passage percolation

Lingfu Zhang
(Joint work with Allan Sly)

Princeton University
Department of Mathematics

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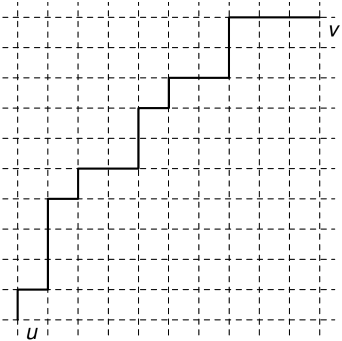
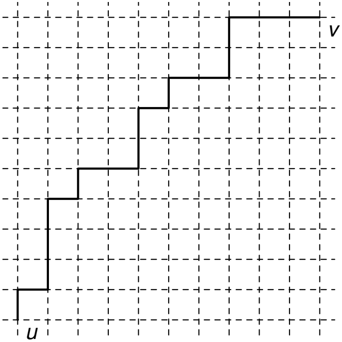
Exactly solvable LPP: model and main results



We study the directed last passage percolation (LPP) on $\mathbb{Z}^2$.

- $\xi(v) \sim \text{Exp}(1)$, i.i.d. $\forall v \in \mathbb{Z}^2$
- Passage time: $X_{u,v} := \max_\gamma \sum_{w\in\gamma} \xi(w)$
- Geodesic: $\Gamma_{u,v} := \text{argmax}_\gamma \sum_{w\in\gamma} \xi(w)$



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Equivalent to TASEP, exactly solvable with $1:2:3$ scaling.

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  $\mathcal{A}_2$ is absolute continuous with respect to Brownian motion (Corwin and Hammond, 2014).
- General initial data: KPZ fixed point (Matetski, Quastel, and Remenik, 2017).

$$x \mapsto n^{-1/3}\left(\sup_y f(y) + X_{(-y,y),(n-x(2n)^{2/3},n+x(2n)^{2/3})} - 4n\right)$$

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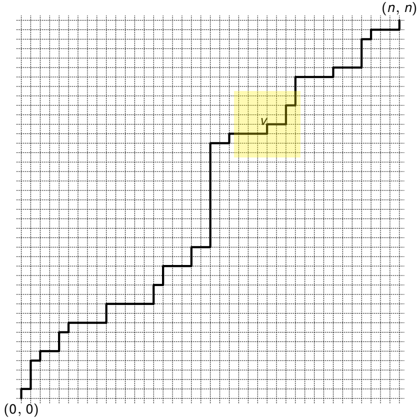

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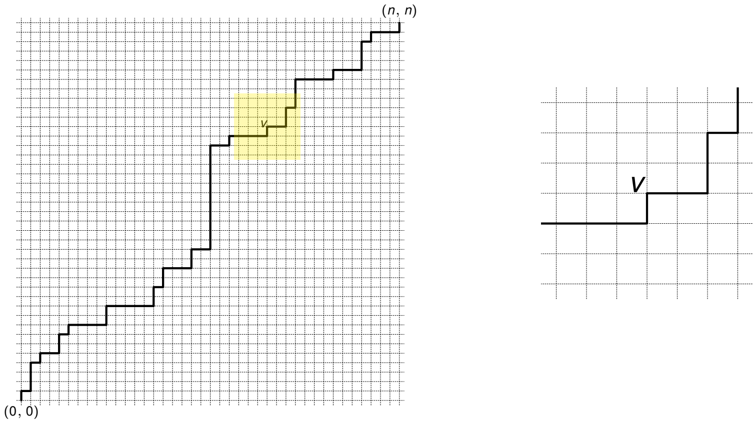

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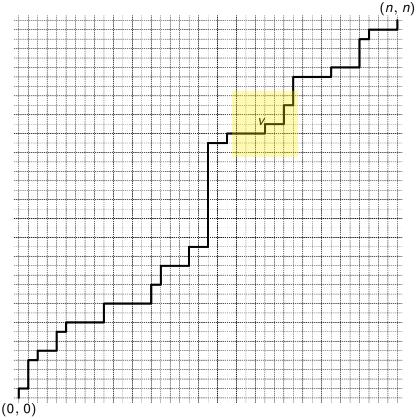

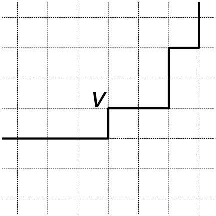

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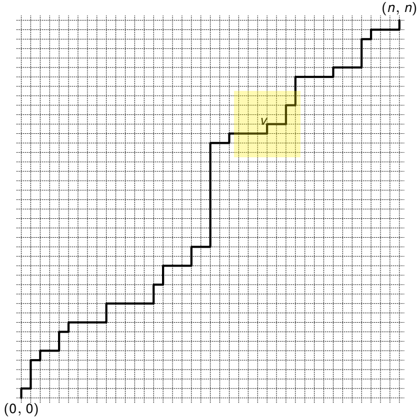

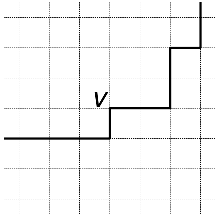

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Main result

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Theorem (Sly and Z., 2020)

For each $s \in \mathbb{Z}_{\geq 0}$, there exists a (deterministic) measure μ_s on $\mathbb{R}^{(2s+1)^2}$, such that $\mu_{n,s} \to \mu_s$ weakly in probability as $n \to \infty$.

Ingredients of the proof

General idea

Weights along the geodesic are asymptotically i.i.d.

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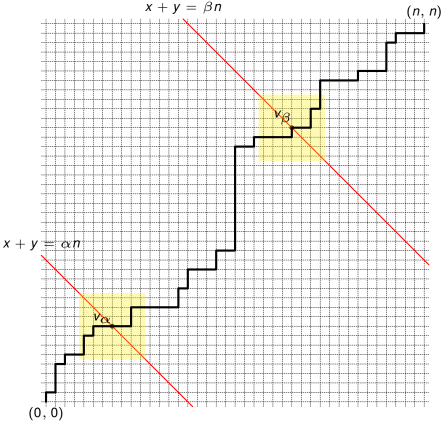

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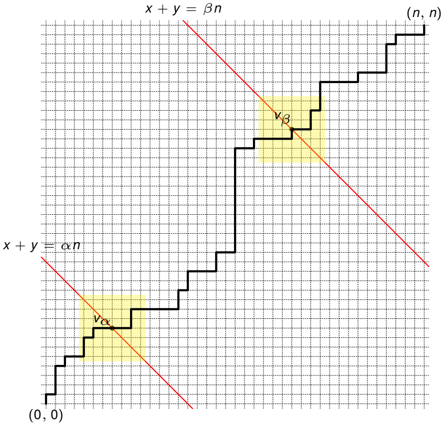

- Find some $\Psi_{n,s}$, s.t. $\forall \alpha, \beta$, as $n \to \infty$, the joint law of $\xi_s(v_\alpha), \xi_s(v_\beta)$ is close to $\Psi_{n,s} \times \Psi_{n,s}$.

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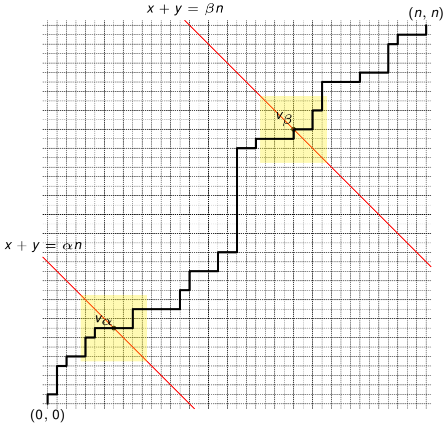

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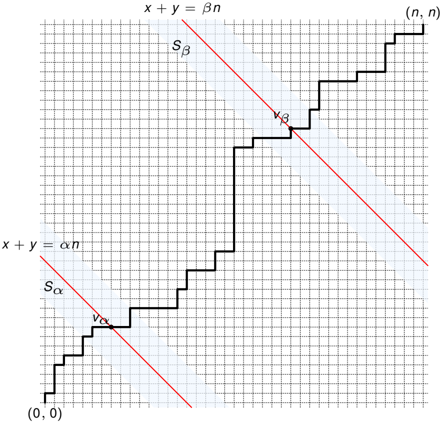
- $\Psi_{n,s}$ converges as $n \to \infty$.

Mostly depends on a strip



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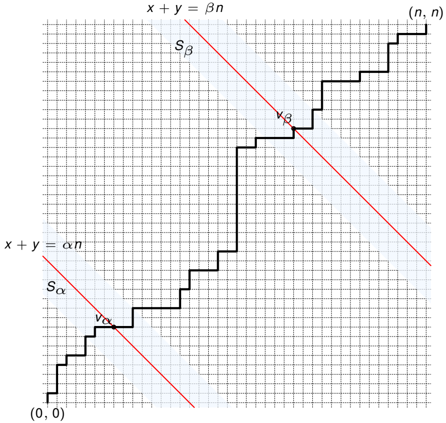

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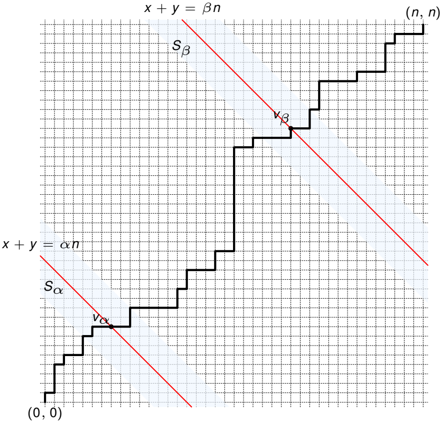

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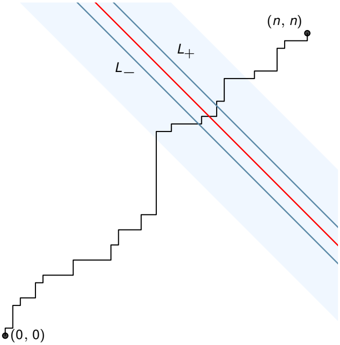
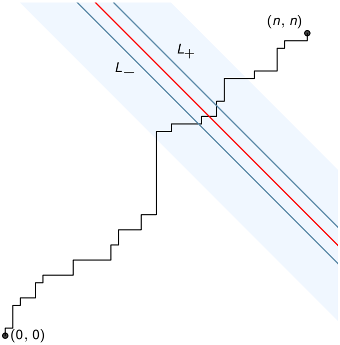
S_α being disjoint from $S_\beta \implies$ asymptotic independence.

A closer look at the strip





- Take L_-, L_+ being δn away from $x + y = \alpha n$.
- Consider the passage times from $(0, 0)$ to L_- and from (n, n) to L_+: $\mathcal{H}_-, \mathcal{H}_+$.
- $X_{(0,0),(n,n)} = \max_{u \in L_-, w \in L_+} X_{u,w} + \mathcal{H}_-(u) + \mathcal{H}_+(w)$. Geodesic between L_- and L_+: $\Gamma_{\mathcal{H}_-,\mathcal{H}_+}$.

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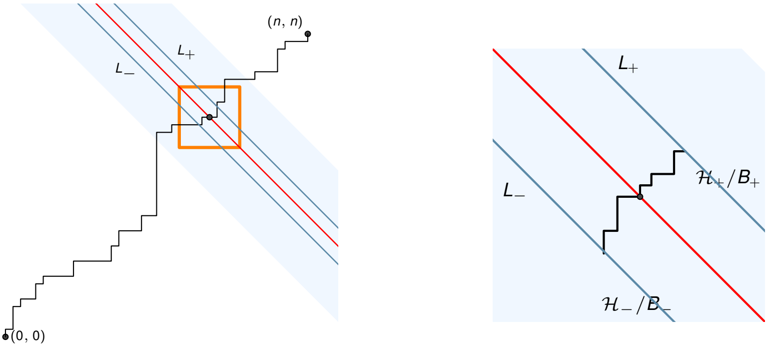

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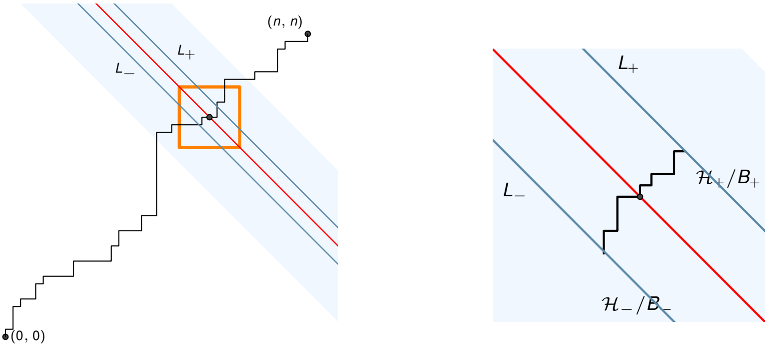
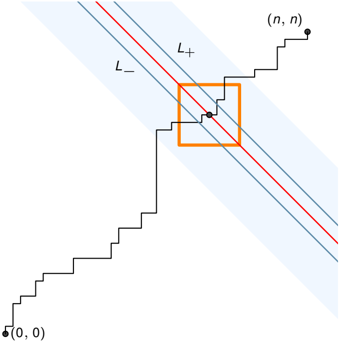
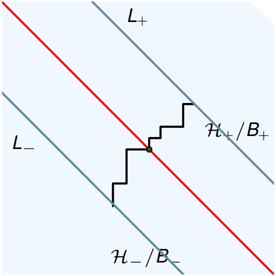


- Conditioned on $\xi(v)$ for $v \notin S_\alpha$, $\mathcal{H}_-, \mathcal{H}_+$ are locally Brownian.
- Around $\operatorname{argmax} \mathcal{H}_- + \mathcal{H}_+$, (with rescaling) the law of $\mathcal{H}_-, \mathcal{H}_+$ is close to B_-, B_+, where $B_- + B_+$ is 3D-Bessel and $B_- - B_+$ is Brownian motion.
- Using KPZ fixed point formulae.





- Replace $\mathcal{H}_-, \mathcal{H}_+$ by B_-, B_+.
- With high prob Γ_{B_-,B_+} largely overlaps with $\Gamma_{\mathcal{H}_-,\mathcal{H}_+}$.
- With high prob $v_\alpha = \Gamma_{\mathcal{H}_-,\mathcal{H}_+} \cap \{x + y = \alpha n\} = \Gamma_{B_-,B_+} \cap \{x + y = \alpha n\}$.

Convergence of $\Psi_{n,s}$

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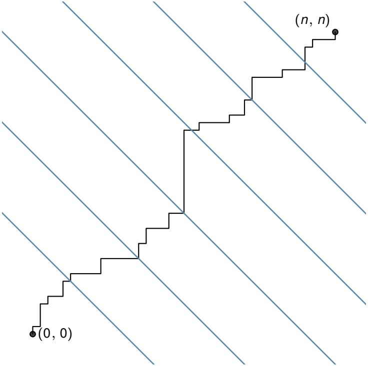
Cover geodesics by length $\sim m$ geodesics, for large fixed m.



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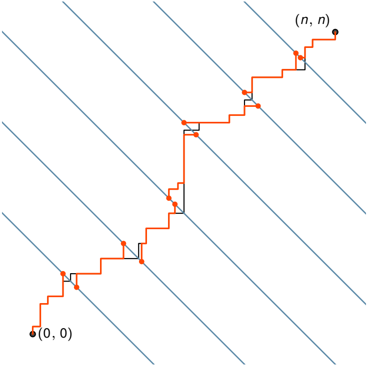

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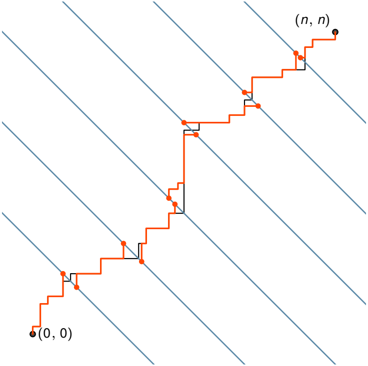

A $1 - \epsilon$ portion of vertices in $\Gamma_{(0,0),(n,n)}$ are covered $\implies \Psi_{n,s}$ is close to $\Psi_{m,s}$.

Thank you!

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