Quenched Multiscale Renormalization

Online Open Probability School (OOPS) 2021

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Based on a joint work with Hilário, Sá and Sanchis

Overview of the course

1. Renormalization in Percolation
2. Quenched renormalization:
   good and bad boxes
3. Quenched renormalization:
   intensity of defects

Renormalization in Percolation

Overview of this lecture

Why renormalization in percolation?

Why renormalization?

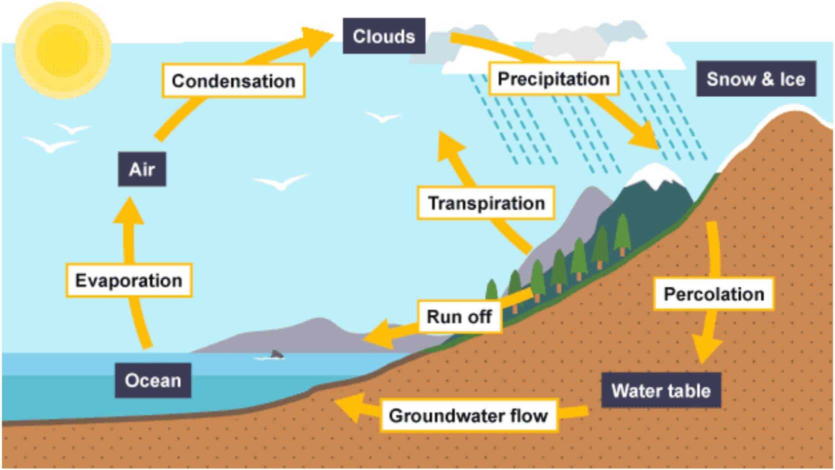
- Very powerful technique
- Make intuitive descriptions rigorous
- Applies to many models
- It is pretty

Why percolation?

- Simple model
- Full of interesting phenomena
- Nice open questions
- Excellent testbed for renormalization

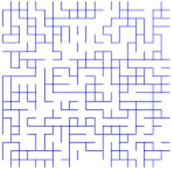
Harry Kesten

Percolation



Bernoulli percolation

- Introduced by Broadbent and Hammerley in 1957.
- Very simple model.
- Extensively studied.
- Fundamental open questions.

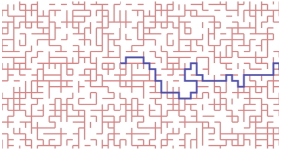


- Consider $\mathbb{Z}^2$ with edges between nearest neighbors.
- Fix $p \in [0, 1]$.
- Every edge is declared open with probability p and closed w.p. $(1 - p)$.
- This is done independently for every edge.

Phase transition

Consider:

$$[0 \leftrightarrow \infty] := \text{there exists an open path from 0 to infinity.} \tag{1}$$



Its probability $\theta(p)$ is weakly monotone in p:

$$\theta(p) := P[0 \leftrightarrow \infty] \tag{2}$$

A beautiful path-counting argument (Peierls) shows that:

- $\theta(p) = 0$ for p small;
- $\theta(p) > 0$ for p close to one.

Phase transition!

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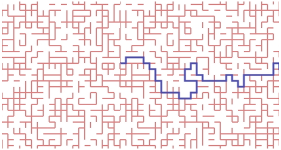

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- $\theta(p) > 0$ for p close to one.

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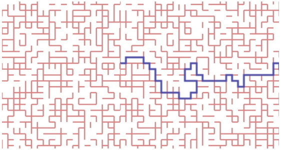

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A beautiful path-counting argument (Peierls) shows that:

- $\theta(p) = 0$ for p small; $\leftarrow$ **We will prove this. And more!**
- $\theta(p) > 0$ for p close to one.

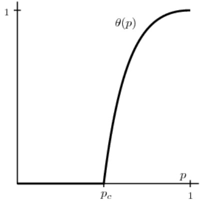
Phase transition!

Open questions

Define $p_c = \sup\{p \in [0,1]; \theta(p) = 0\}$.

(Harris + Kesten) proved that for $\mathbb{Z}^2$:

- $p_c = 1/2$;
- $\theta(p)$ is continuous in p.



There are still many question that remain open concerning this model:

- Is $\theta(p)$ continuous for dimensions $3, 4, ..., 10$?
- How does $\theta(p)$ behave as p approaches p_c?

Multi-scale Renormalization

What we are going to prove?

Theorem

There exists $p_0 \in (0,1)$ such that for $p \leq p_0$

$$\mathbb{P}_p\Big[0 \leftrightarrow \infty\Big] = 0.$$

Actually

$$\mathbb{P}_p\Big[0 \leftrightarrow \partial B_n\Big] \leq \exp\{-n^{0.1}\},$$

for all $n \geq 1$.

Multi-scale Renormalization

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$$\mathbb{P}_p\Big[0 \leftrightarrow \partial B_n\Big] \leq \exp\{-n^{0.1}\},$$

for all $n \geq 1$.

Obs:

- Counting paths are easier and give better bounds (p_0 and on decay)
- Renormalization is much more robust

Outline of the proof

Steps of the proof:

A) Chose scales
B) Define “bad event”*
C) Prove “cascading property”
D) Recursive inequalities**
E) Perform triggering

*Looks easy but it is hard

**Looks hard but it is easy

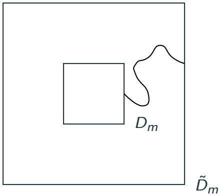
Step A (Choose scales)

Let $L_k = 9^k$, for $k \geq 0$.

$M_k = \{k\} \times \mathbb{Z}^2$.

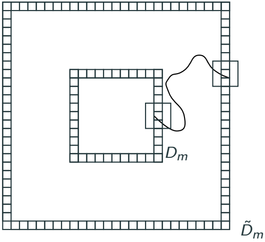
Also $\{D_m\}_{m \in M_k}$ is a paving of $\mathbb{Z}^2$ with boxes of side L_k.

Step B (Define bad events)



$$p_k = \mathbb{P}(E_m), \text{ for some } m \in M_k.$$

Step C (Cascading Property)

$E_m =$ 

If $m \in M_{k+1}$,

$$E_m \subseteq \bigcup_{m_1, m_2} E_{m_1} \cap E_{m_2}, \qquad \text{with } m_1, m_2 \in M_k.$$

Consequently

$$p_{k+1} \leq 27^4 p_k^2.$$

Step D (Recursive inequalities)

We want to prove that

$$p_k \leq \exp\left\{-L_k^{0.1}\right\}, \qquad \text{for every } k \geq 0.$$

Induction step - Suppose true for k:

$$\begin{aligned}
\frac{p_{k+1}}{\exp\{-L_{k+1}^{0.1}\}} &\overset{\text{Cascading}}{\leq} \frac{1}{\exp\{-L_{k+1}^{0.1}\}} 27^4 p_k^2 \\
&\overset{\text{Induction}}{\leq} \frac{1}{\exp\{-L_{k+1}^{0.1}\}} 27^4 \exp\{-2L_k^{0.1}\} \\
&= 27^4 \exp\left\{-\left(2L_k^{0.1} - L_{k+1}^{0.1}\right)\right\} \\
&= 27^4 \exp\left\{-\left(2L_k^{0.1} - 9^{0.1} L_k^{0.1}\right)\right\} \\
&\overset{k \geq k_o}{\leq} 1,
\end{aligned}$$

since $9^{0.1} \sim 1.24\ldots$

Step E (Triggering)

Still need for some $k \geq k_o$

$$p_k \leq \exp\{-L_k^{0.1}\}. \tag{3}$$

Pick p small enough.

Conclusion

$$\mathbb{P}[0 \leftrightarrow \infty] \leq \mathbb{P}[B_{L_k} \leftrightarrow \partial 3B_{L_k}] \leq \exp\{-L_k^{0.1}\} \underset{k}{\longrightarrow} 0.$$

Advantages

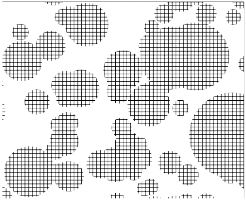
- Not restricted to percolation
- Quantitative results
- Robust to microscopic changes
- Robust to dependence
- Implicit condition (3).

Review

Steps of the proof:

A) Chose scales
B) Define “bad event”
C) Prove “cascading property”
D) Recursive inequalities
E) Perform triggering

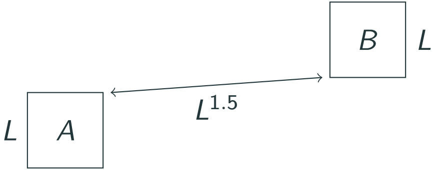
Dependent percolation



Model:

- $\{X_i\}_{i\geq 0}$ is a PPP with intensity u
- $\{R_i\}_{i\geq 0}$ i.i.d. radii $P[R_i > r] \leq r^{-20}$
- Add edges inside $B(X_i, R_i)$

Percolation is dependent, but satisfies



$$\mathbb{P}(A \cap B) \leq \mathbb{P}(A)\mathbb{P}(B) + L^{-10}. \qquad (\star)$$

Step A (Choose scales)

Let $L_0 = 100$,

$$L_{k+1} \sim L_k^{1.5} \qquad \text{(actually } \lfloor L_k^{0.5} \rfloor L_k\text{)}$$

Entropy problem?

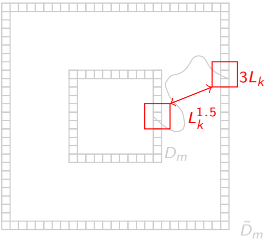
$M_k = \{k\} \times \mathbb{Z}^2$.

Also $\{D_m\}_{m \in M_k}$ is a paving of $\mathbb{Z}^2$ with boxes of side L_k.

Step B (Define bad events)

$$p_k = \mathbb{P}(E_m), \text{ for some } m \in M_k.$$

Step C (Cascading Property)

$E_m =$ 

Consequently

$$p_{k+1} \leq \left(\frac{3L_{k+1}}{L_k}\right)^4 \sup_{m_1,m_2} \mathbb{P}\big(E_{m_1} \cap E_{m_2}\big)$$
$$\leq 3^4 L_k^2 \Big(p_k^2 + L_{k+1}^{-10}\Big).$$

Step D (Recursive inequalities)

We want to prove that

$$p_k \leq L_k^{-8}, \qquad \text{for every } k \geq 0.$$

Induction step - Suppose true for k:

$$\begin{aligned}
\frac{p_{k+1}}{L_{k+1}^{-8}} &\overset{\text{Cascading}}{\leq} \frac{1}{L_{k+1}^{-8}} 3^4 L_k^2 \Big(p_k^2 + L_k^{-10} \Big) \\
&\overset{\text{Induction}}{\leq} 3^4 L_{k+1}^8 L_k^2 \Big(L_k^{-16} + L_{k+1}^{-10} \Big) \\
&= 3^4 L_k^{12+2} \Big(2 L_k^{-15} \Big) \\
&\overset{k \geq k_o}{\leq} 1,
\end{aligned}$$

since $15 > 14$.

Step E (Triggering)

Still need for some $k \geq k_o$

$$p_k \leq \exp\{-L_k^{0.1}\}. \tag{4}$$

Pick u small enough.

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Approximate independence is uniform over $u \leq 1$!!!

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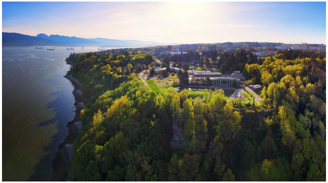
Approximate independence is uniform over $u \leq 1$!!!

Conclusion

$$\mathbb{P}[0 \leftrightarrow \infty] \leq \mathbb{P}[B_{L_k} \leftrightarrow \partial 3B_{L_k}] \leq L_k^{-8} \underset{k}{\longrightarrow} 0.$$

Thank you!



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Quenched renormalization: good and bad boxes

Overview of this lecture

2 Quenched renormalization:
good and bad boxes

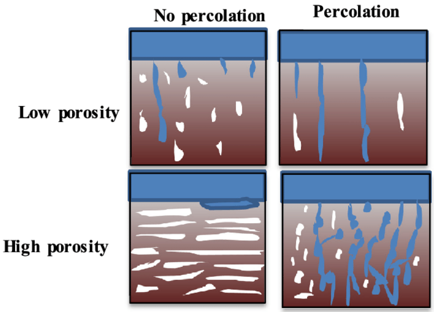
- Columnar defects
- Negative results
- Environment: Good-box, Bad-box
- Percolation
- What comes next

What are we exercising?

Let us flex our technique:

- Quenched renormalization
- Crazy scales
- Crazy cascading property

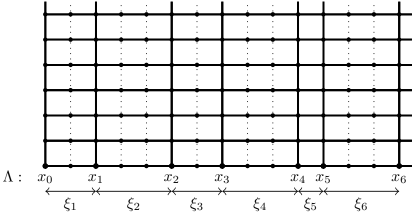
Inhomogeneous Percolation



Difficulty to represent different media.

A different model

(a) A typical layered rock



(b) A new model

Our model for graph G:

- The set of vertices of G is $\mathbb{Z}_+^2$;
- **Horizontal nearest neighbor edges:** add them all;
- Given integers $0 = x_0 < x_1 < x_2 < \dots$
- **Vertical nearest neighbor edges:**
  add the ones that lie in some line $\{x_i\} \times \mathbb{R}$, for $i \geq 0$.

How we chose x_i's?

Pick $\xi_1, \xi_2, \ldots$ i.i.d integer random variables (tail of defects).

Let

$$X_i = \sum_{i=1}^{i} \xi_i \tag{5}$$

This is a Renewal Process.

Observation

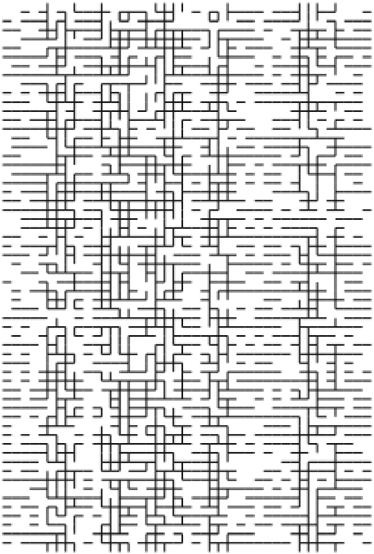
It is clear that our graph G is a subgraph of $\mathbb{Z}^2$ (with n.n. edges)

Therefore, for $p \leq 1/2$ we have $\theta(p) = 0 \quad$ (thus $p_c \geq 1/2$).

Question

- Is $p_c < 1$ (phase transition)?
- How the above question depends on the distribution of ξ?

Simulation



Previous work

Theorem (Bramson, Durrett, Schonmann)

Suppose that there is some $c > 0$ such that

$$P(\xi_i > k) \leq e^{-ck}, \text{ for every } k \text{ large enough,} \tag{6}$$

then $p_c < 1$ for a.e. realization of X_i's.

Observations

- BDS was originally stated for the contact process.
- Our article is very inspired by BDS (questions and proof).
- Hoffman: horizontal lines removed as well (more on that later).
- Kensten, Sidoravicius, Vares: oriented case.
- Duminil-Copin, Hilário, Kozma, Sidoravicius: near-critical.

Main results

Theorem (Hilário, Sá, Sanchis, T.)

Suppose that for some $\eta > 1$ we have $E(\xi^\eta) < \infty$. Then $p_c < 1$ for a.e. realization of X_i's.

Theorem (Hilário, Sá, Sanchis, T.)

Suppose that for some $\eta < 1$ we have $E(\xi^\eta) = \infty$. Then $p_c = 1$ for a.e. realization of X_i's.

Observations

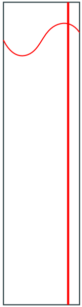
- Interpreting the "thickness of defects".
- What happens if $E(\xi) = \infty$?
- What if $E(\xi) < \infty$?

Absence of percolation

Suppose $E(\xi^\eta) = \infty$ for some $\eta < 1$.

Fixing $\eta < \eta' < 1$, consider the rectangle

$$\big[0, i\big) \times \big[0, \exp\{i^{1/\eta'}\}\big). \tag{7}$$

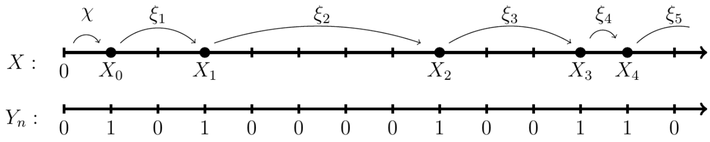


With reasonable probability:

- There will be some $\xi_i > i^{1/\eta}$.
- The percolation will not survive this long corridor.

End with Borel-Cantelli.

An alternative definition



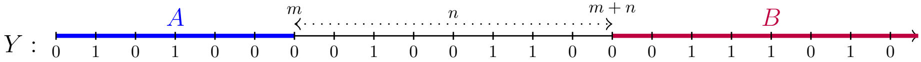
We can alternatively study

$$Y_n = \mathbf{1}\{X_i = h; \text{for some } i\}, \text{ for } n \geq 0. \tag{8}$$

We then change the first jump to χ to make the renewal process stationary:

$$(Y_0, Y_1, \dots) \overset{d}{\sim} (Y_l, Y_{l+1}, \dots). \tag{9}$$

Decoupling



Lemma

Let $\xi_i \geq 1$ be an i.i.d, aperiodic, integer-valued sequence of increments satifying

$$E(\xi^{1+\epsilon}) < \infty, \text{ for some } \eta > 1. \tag{10}$$

Then, there is $c = c(\xi, \epsilon)$ such that for any pair of events

$$A \in \sigma(Y_i; 0 \leq i \leq n) \quad \text{and} \quad B \in \sigma(Y_i; i \geq m+n), \tag{11}$$

we have that

$$P(A \cap B) = P(A)P(B) \pm cn^{-\epsilon}. \tag{12}$$

Recall our 5 steps!!!

Steps of the proof:

A) Chose scales
B) Define “bad event”
C) Prove “cascading property”
D) Recursive inequalities
E) Perform triggering

Multiscale renormalization

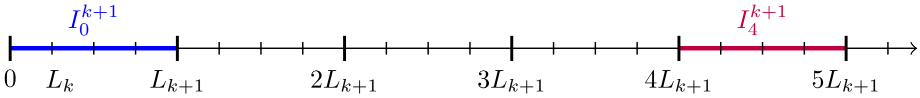
Choosing appropriately $L_0 \geq 1$ and $\gamma > 1$ we define

$$L_{k+1} = L_k \lfloor L_k^{\gamma-1} \rfloor \sim L_k^{\gamma}, \text{ for } k \geq 1. \tag{13}$$

We also pave $\mathbb{Z}_+$ with the intervals

$$I_j^k = [jL_k, (j+1)L_k), \text{ for } j \geq 0 \tag{14}$$

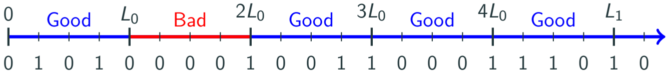
Cover I_j^{k+1} with blocks at scale k



Good and Bad intervals

Step B (Define bad events)

Scale 0: no good column.



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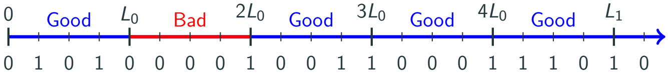
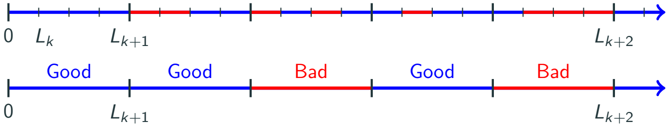


Scale $k+1$**:** two **non-consecutive** bad blocks at scale k.



Typical boxes are good

Define

$$p_k := P\left[I_k \text{ is } \textit{bad}\right]$$

Lemma

There exists $\alpha > 0$ *such that*

$$p_k \leq L_k^{-\alpha}, \tag{15}$$

for every $k \geq 0$.

Typical boxes are good

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Step C (Cascading Property) ✓

Step D (Recursive inequalities)

We want to prove that

$$p_k \leq L_k^{-\alpha}, \qquad \text{for every } k \geq k_o.$$

Induction step - Suppose true for k:

$$\begin{aligned}
\frac{p_{k+1}}{L_{k+1}^{-\alpha}} &\overset{\text{Cascading}}{\leq} \frac{1}{L_{k+1}^{-\alpha}} \left(\frac{L_{k+1}}{L_k} \right)^2 \sup_{m_1, m_2} P\Big[\mathrm{Bad}(m_1) \cap \mathrm{Bad}(m_2) \Big] \\
&\leq L_k^{2(\gamma-1)+\gamma\alpha} \Big(p_k^2 + L_k^{-\epsilon} \Big) \\
&\overset{\text{Induction}}{\leq} 2 L_k^{2(\gamma-1)+\gamma\alpha - 2\alpha \wedge \epsilon} \\
&\overset{k \geq k_o}{\leq} 1,
\end{aligned}$$

since we pick $2\alpha < \epsilon$ and $2 - \gamma > \frac{2(\gamma-1)}{\alpha}$.

Step E (Triggering)

Choose L_0 large.

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It is actually tricky because k_0 grows !!!

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Conclusion

$$\mathbb{P}[I_k \text{ is } \mathit{bad}] = p_k \le L_k^{-\alpha} \underset{k}{\longrightarrow} 0.$$

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Choose L_0 large.

It is actually tricky because k_0 grows !!!

Conclusion

$$\mathbb{P}[I_k \text{ is } \mathit{bad}] = p_k \leq L_k^{-\alpha} \underset{k}{\longrightarrow} 0.$$

Now we need to deal with percolation.

Thus the name **"Quenched Renormalization"**.

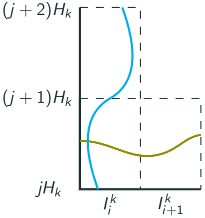
Step A (Choose scales)

Vertical scales (fix $\mu \in (\frac{1}{\nu}, 1)$)

$$H_0 = 100 \quad \text{and} \quad H_{k+1} = 2\lceil \exp(L_{k+1}^{\mu}) \rceil H_k, \qquad \text{for } k \geq 0. \tag{16}$$

Step B (Define bad events)

Crossing events: C_m and D_m



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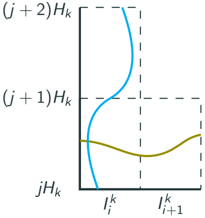


$$r_k := \max_{\substack{\Lambda;\ I_i^k, I_{i+1}^k \\ \text{good}}} \mathbb{P}_p^{\Lambda}\big((C_m)^c\big) \qquad s_k := \max_{\substack{\Lambda;\ I_i^k \\ \text{good}}} \mathbb{P}_p^{\Lambda}\big((D_m)^c\big).$$

We want to prove

Lemma

There exists $p_0, k_0, \beta > 0$ *such that for* $p > p_0$

$$\max\{r_k, s_k\} \leq \exp\Big\{ - L_k^{\beta} \Big\}, \text{ for all } k \geq k_0.$$

We want to prove

Lemma

There exists $p_0, k_0, \beta > 0$ *such that for* $p > p_0$

$$\max\{r_k, s_k\} \leq \exp\Big\{-L_k^\beta\Big\}, \text{ for all } k \geq k_0.$$

We actually do this in two steps:

Lemma (R-Lemma)

If $\max\{r_k, s_k\} \leq \exp\{-L_k^\beta\}$*, then*

$$r_{k+1} \leq \exp\Big\{-L_{k+1}^\beta\Big\}.$$

Lemma (S-Lemma)

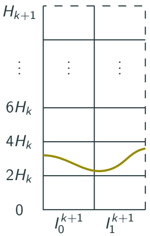
If $\max\{r_k, s_k\} \leq \exp\{-L_k^\beta\}$*, then*

$$s_{k+1} \leq \exp\Big\{-L_{k+1}^\beta\Big\}.$$

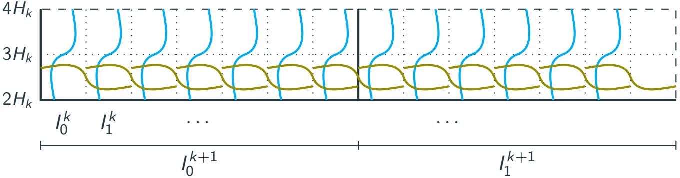
R-Lemma

Step C (Cascading Property)

If C_m fails, no crossing in any corridor ($\exp\{L^\mu_{k+1}\}$ many of them).



If a corridor is not crossed, one event below fails



Step D (Recursive inequalities) for R-Lemma

Induction step -

$$
\begin{aligned}
\frac{r_{k+1}}{\exp\{-L_{k+1}^{\beta}\}} &\overset{\text{Cascading}}{\leq} \frac{1}{\exp\{-L_{k+1}^{\beta}\}}\Big(1-\big(1-r_k\big)^{L_k^{\gamma-1}}\big(1-s_k\big)^{L_k^{\gamma-1}}\Big)^{\exp\{L_{k+1}^{\mu}\}} \\
&\overset{\text{Induction}}{\leq} \exp\{L_{k+1}^{\beta}\}\Big(1-\big(1-2L_k^{\gamma-1}\exp\{-L_k^{\beta}\}\big)\Big)^{\exp\{L_{k+1}^{\mu}\}} \\
&= \exp\{L_{k+1}^{\beta}\}\big(2L_k^{\gamma-1}\exp\{-L_k^{\beta}\}\big)^{\exp\{L_{k+1}^{\mu}\}} \\
&\overset{k\text{ large}}{=} \exp\{L_{k+1}^{\beta}\}2^{-\exp\{L_{k+1}^{\mu}\}} \\
&\overset{k\geq k_o}{\leq} 1,
\end{aligned}
$$

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That was easy, right !?!?

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\end{aligned}
$$

That was easy, right !?!?

Sorry

What was wrong?

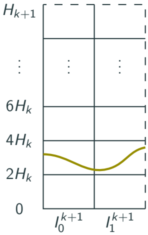
We forgot the bad boxes.

$$r_k := \max_{\substack{\Lambda;\ I_i^k, I_{i+1}^k \\ \text{good}}} \mathbb{P}_p^\Lambda\big((C_m)^c\big)$$

A good box $k+1$ can have two bad boxes inside!

Second chance

If C_m fails, no crossing in any corridor ($\exp\{L_{k+1}^{\mu}\}$ many of them).



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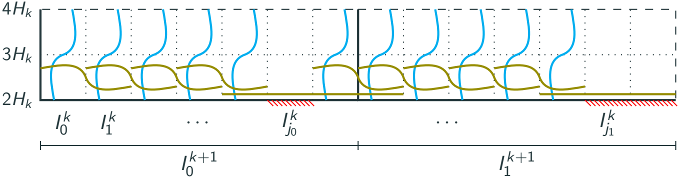


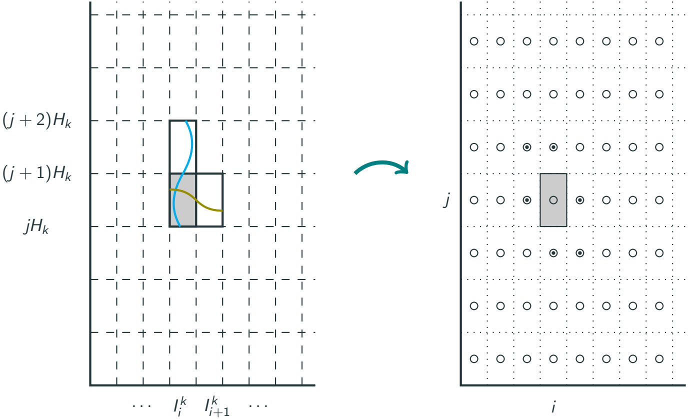
Step D (Recursive inequalities) for R-Lemma

Induction step -

$$\begin{aligned}
\frac{r_{k+1}}{\exp\{-L_{k+1}^{\beta}\}} &\overset{p>1/2}{\leq} \exp\{L_{k+1}^{\beta}\}\left(1 - 2^{-8L_k - 1}\right)^{\exp\{L_{k+1}^{\mu}\}} \\
&= \exp\left\{L_{k+1}^{\beta} - 2^{-8L_k - 1} e^{L_{k+1}^{\mu}}\right\} \\
&= \exp\left\{L_{k+1}^{\beta} - 2^{-8L_k - 1} e^{L_{k}^{\mu\gamma}}\right\} \\
&\overset{k \text{ large}}{\leq} 1,
\end{aligned}$$

because $\gamma\mu > 1$.

Step C (Cascading Property)



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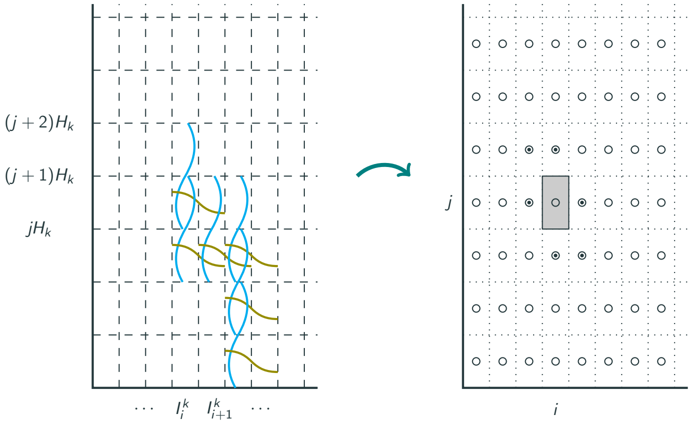
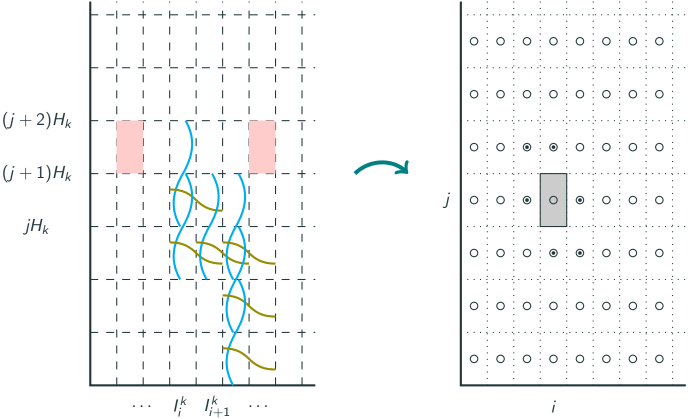


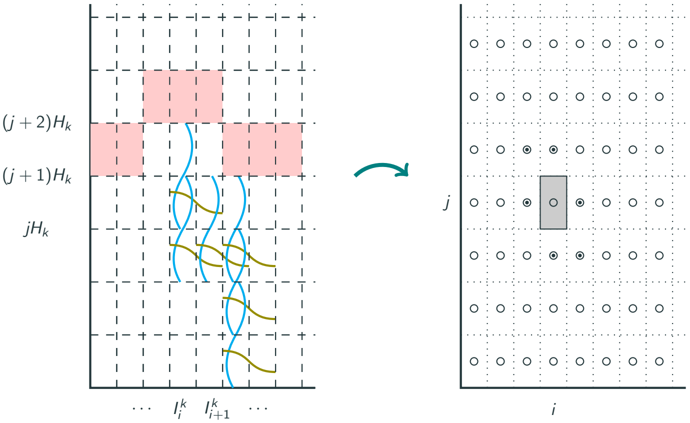
S-Lemma

Step C (Cascading Property)



S-Lemma

Step C (Cascading Property)



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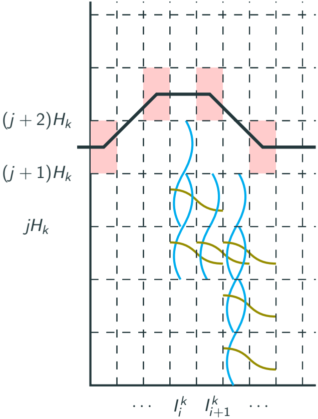
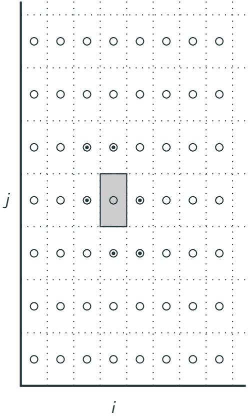




Step D (Recursive inequalities) for S-Lemma

Induction step -

$$\frac{s_{k+1}}{\exp\{-L_{k+1}^{\beta}\}} \overset{\text{Cascading}}{\leq} e^{L_{k+1}^{\beta}} \sum_{n} P\big[\text{“dashed”, blocking path of length } n\big]$$

$$\overset{\text{Induction}}{\leq} e^{L_{k+1}^{\beta}} \sum_{n \geq L_k^{\gamma-1}} \underbrace{\exp\{L_{k+1}^{\mu}\}}_{\text{starting point}} \quad \underbrace{8^n}_{\#\text{ of paths}} \quad \underbrace{\exp\{-L_k^{\beta}\}^{n/7}}_{\text{probability of path}}$$

$$\leq \exp\left\{L_{k+1}^{\beta} + L_k^{\gamma\mu}\right\} \sum_{n \geq L_k^{\gamma-1}} 8^n \exp\{-L_k^{\beta}\}^{n/7}$$

$$\leq C \exp\left\{L_k^{\beta\gamma} + L_k^{\gamma\mu}\right\} 8^{L_k^{\gamma-1}} \exp\{-L_k^{\beta} L_k^{\gamma-1}/7\}$$

$$\overset{k \geq k_o}{\leq} 1,$$

since $\beta + \gamma - 1 > \max\{\gamma\beta, \gamma\mu\}$ $\quad$ ($\beta < 1$, but close).

Final comments

Where to go next?

- Good/bad boxes are well suited for large defects
- Use up a lot of vertical space
- If we remove horizontal lines, the argument breaks

Thank you!

100

Quenched renormalization: intensity of defects

Recalling last lecture

In the last lecture:

- Defects on x-axis only,
- Large defects,
- Defects could be considered catastrophic,
- Needed a lot of vertical room.

Overview of this lecture

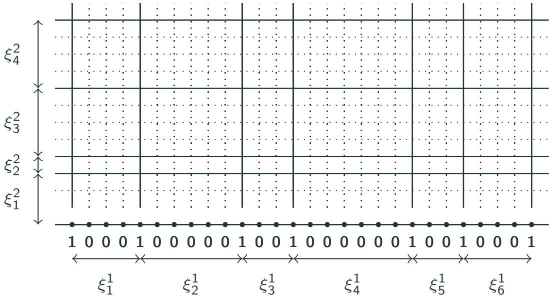
3 Quenched renormalization:
intensity of defects

- Model
- Environment: Intensity of defects
- Percolation: good boxes
- How to cross a trap

The definition of the model

The model

- Two sequences ξ^1, ξ^2 of i.i.d. Geo(ρ) random variables
- Stretch the lattice horizontally (by ξ^1) and vertically (by ξ^2)



Perform Bernoulli percolation p on this stretched lattice.

Simulation

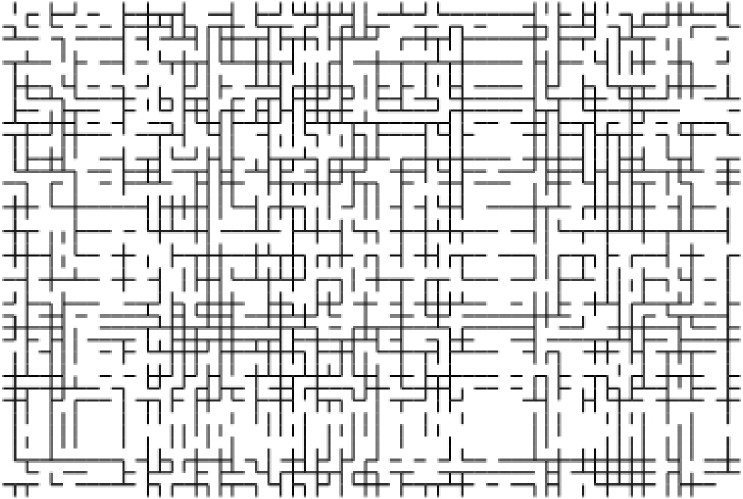


Figure 4

History of the problem

Conjecture 2000

[Jonasson, Mossel, Peres] For $\rho > 0$ small and $p < 1$ large, there is percolation.

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Answer 2005

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Theorem (Hoffman)

There exists $\rho > 0$ and $p < 1$ such that

$$\mathbb{P}_p^\rho[0 \leftrightarrow \infty] > 0.$$

Hoffman's proof follows a dynamic renormalization.

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Theorem (Hoffman)

There exists $\rho > 0$ and $p < 1$ such that

$$\mathbb{P}_p^\rho[0 \leftrightarrow \infty] > 0.$$

Hoffman's proof follows a dynamic renormalization.

We will sketch a proof of this result using a static renormalization.

Very inspired by Hoffman.

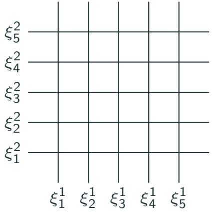
Outline of the proof

Here is a quick guide

- Our 5-step guide to success for the environment
- Our 5-step guide to success for percolation on good boxes
- How to traverse obstacles

Important observation:

- We look at the values of ξ_i only
- ξ_i^1 refers to "east edge"
- ξ_i^2 refers to "north edge"
- Edge is open with probability p^{ξ_i+1}



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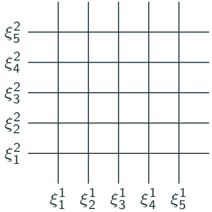


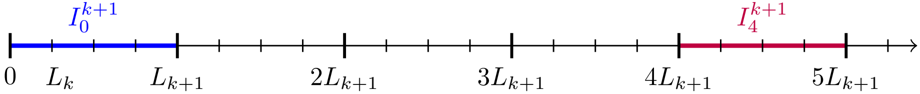
Step A (Choose scales)

$$L_k = 500^k, \qquad \text{for } k \geq 0.$$

As before

$$I_j^k = [j500^k, (j+1)500^k) \cap \mathbb{Z}$$

These are nested intervals



Environment

We want to “grade” defects:

- For each interval I_j^k, we associate a defect $H_j^k = 0, 1, \ldots$
- An interval with $H = 0$ is called **good**, otherwise **bad**.

Scale 0 -

- $L_0 = 1$,
- $I_j^k = \{j\}$,
- $H_j^k = \xi_j$.

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Scale $k+1$ -

$$H_j^{k+1} = \begin{cases} 0, & \text{if all sub-intervals are good} \\ H_{j_o}^k - 1, & \text{if } j_o \text{ is the only bad sub-interval} \\ \sum_{l=0}^{L} H_{j_l}^k + 20L & \text{if } j_0, \dots, j_L \text{ are the bad intervals} \end{cases}$$

Step C (Cascading Property) ✓

Define

$$p_k = \mathbb{P}[I_j^k \text{ is bad}] = \mathbb{P}[H_0^k \geq 1].$$

Lemma

For ρ small enough

$$p_k \leq L_k^{-10},$$

for every $k \geq 0$.

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And other simplifications along the way!

We actually prove

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$$\mathbb{P}\left[H_0^k = h\right] \leq 500^{-10k-20h}$$

Trying to prove more makes it easier (induction).

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Scale 0 -

$$\mathbb{P}\left[H_0^0 = h\right] = \mathbb{P}[\xi_0 = h] = \rho^h \leq 500^{-20h}.$$

Scale $k+1$ - Roughly

$$\mathbb{P}\left[H_0^{k+1} = h\right] \leq \sum_{\substack{h_0,\dots,h_L; \\ h=\sum h_l - 20L}} \prod_{l=0}^{L} 500^{-10k-20h_l}$$

$$\leq \cdots \leq 500^{-10k-20h}.$$

Good rectangles

Retangles

$$R_{i,j}^{k} = \left[iL_k, (i+1)L_k\right) \times \left[jL_k, (j+1)L_k\right)$$

Good rectangles

Retangles

$$R^k_{i,j} = \big[iL_k, (i+1)L_k\big) \times \big[jL_k, (j+1)L_k\big)$$

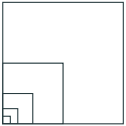
We call them **good** if

$$H^k_i = H^k_j = 0$$

Observation

There exists $\rho > 0$ so that

$$\mathbb{P}\big[R^k_{(0,0)} \text{ is good for all } k \geq 0\big] > 0$$



Just notice that

$$\sum_k L_k^{-10} = \sum_k 500^{-10k} < 1$$

Definition of filled boxes

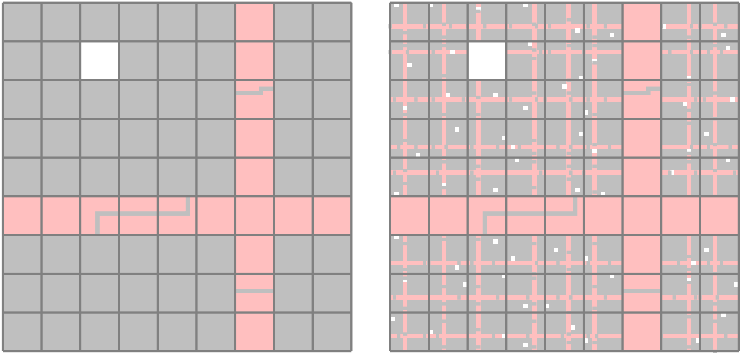
Scale 0

- $L_0 = 1$,
- $R^0_{i,j} = (i, j)$,
- It is filled if its *north* and *east* edges are open,
- $P[R^0 \text{ filled}] = p^2$.
- Its cluster is $\mathcal{C}^k_{i,j} = \{(i, j)\}$

Scale 1

- all good sub-boxes are filled, except for at most one
- all clusters of filled sub-boxes are connected (we call it $\mathcal{C}^k_{i,j}$)

Percolation



A filled box and its cluster $\mathcal{C}_{i,j}^k$ in gray

Define

$$r_k = \sup_{\omega; R_{i,j}^k \text{ is good}} \mathbb{P}\big[R_{i,j}^k \text{ is not filled}\big].$$

Proof of percolation

Lemma

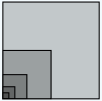
There exists $p < 1$ *such that*

$$r_k \leq 500^{-2k-100}, \quad \textit{for every } k \geq 0.$$

Proof of main theorem.

Assuming the lemma above:

$$\mathbb{P}\left[R^k_{(0,0)} \text{ filled } \forall k \geq 0 \middle| R^k_{(0,0)} \text{ good } \forall k \geq 0\right] > 0.$$



Just notice that $\sum_k r_k < 1$.

□

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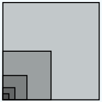

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□

All we need to prove is the lemma!

We really wanted to have

$$r_{k+1} \leq 500^4 r_k^2,$$

but there are bad columns.

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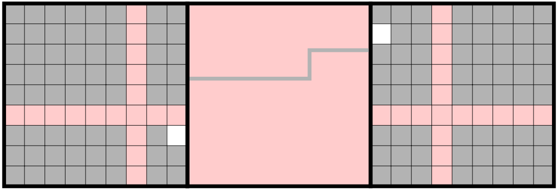
Define

$$s_k = \sup_{\substack{H^k_{(0,0)}=H^k_{(2,0)}=0,\\ H^k_{(1,0)}=1}} \mathbb{P}\Big[\big[\text{either } R^k_{(0,0)} \text{ or } R^k_{(2,0)} \text{ is not filled}\big] \cup \big[\mathcal{C}^k_{(0,0)} \not\leftrightarrow \mathcal{C}^k_{(2,0)}\big]\Big]$$



We call this a “crossing a trap”.

Lemma

Suppose that for $k \geq 0$,

$$r_k \leq 500^{-2k-100} \quad \textit{and} \quad s_k \leq 500^{-2k-80},$$

Then

$$r_{k+1} \leq 500^{-2(k+1)-100}.$$

Proof.

If R^{k+1} is not filled:

- there are two good but non-filled sub-boxes,
- there are two disjoint non-crossed traps.

$$\begin{aligned} r_{k+1} &\leq 500^4 r_k^2 + 1000^2 s_k^2 \\ &\leq 2 \cdot 500^{4-4k-160} \leq 500^{-(k+1)-100}. \end{aligned}$$

□

Crossing defects!

Crossing traps

We need to cross $H = 1$.

Crossing traps

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For this we need to cross all values of H.

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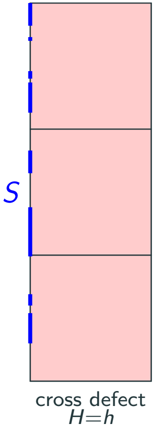

Armies

Definition

S is called regular if

- S only intersects good intervals: $S \cap I_j^k \neq \varnothing \quad \Rightarrow \quad H_j^k = 0$
- S is spread out: S intersects at most 400 sub-intervals of any interval.

Motivation:

- crossing in a bad line is hard.
- packed armies are inefficient.

Observation

Every filled box contains a regular set of size 400^k at its right face.

Intuition

Simple algebraic intuition:

$$\begin{array}{ccc} \text{Regular army} & & \text{becomes regular army} \\ \text{of size } 400^{k+(h-1)/2} & \xrightarrow[\text{defect } H = h]{} & \text{of size } 400^{k} \end{array}$$

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Making this rigorous

$$v_k = \sup_{h,S,\omega} \mathbb{P}\Big[\text{survivors do not contain a regular army of size } 400^k\Big],$$

where the suppremum is taken over

- $h \geq 0$,
- ω such that $H(\text{column}) = h$,
- S regular with $|S| \geq 400^{k+(h-1)/2}$.

Last step of the proof

v_k is stronger than s_k!!!

Control on r_k and v_k $\Rightarrow$ control on r_k and s_k $\Rightarrow$ control on r_{k+1}.

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Control on r_k and v_k $\Rightarrow$ control on r_k and s_k $\Rightarrow$ control on r_{k+1}.

Control over v_k:

Scale 0 -

- Subsets of regular sets are regular,
- Surviving army is Bin $\left(400^{(h-1)/2}, p^{h+1}\right)$,
- If p is large, $P[\text{no survivors}] < 500^{-90}$, for every $h \geq 1$.

Last step of the proof

v_k is stronger than s_k!!!

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Lemma

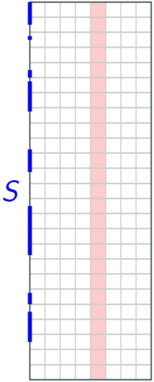
Suppose

$$r_k \leq 500^{-2k-100}, \qquad \textit{and} \qquad v_k \leq 500^{-2k-90},$$

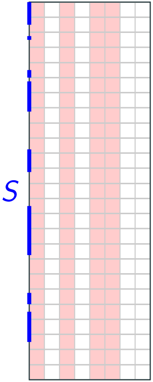
then

$$v_{k+1} \leq 500^{-2k-90}.$$

Two cases to consider



single bad sub-box



multiple bad sub-boxes

Many bad sub-boxes

In this case

$$h = \sum_{l=0}^{L} h_l + 20L.$$

We start with

$$|S| = 400^{k+1+(h-1)/2}$$
$$= 400^{h_0/2+20}$$
$$\times 400^{k+1+(h_1+\cdots+h_L-1)/2+20(L-1)}$$

Many bad sub-boxes

In this case

$$h = \sum_{l=0}^{L} h_l + 20L.$$

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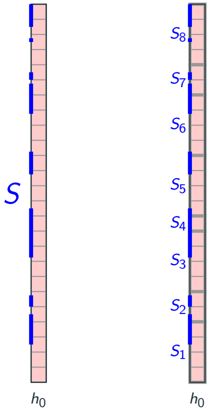

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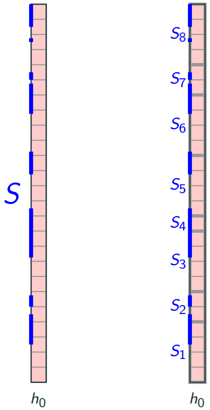

We use v_k and repeat this for each defect.

With high probability we end up with 400^k points.

Single bad sub-box

In this case

$$h = h_0 - 1$$

Then

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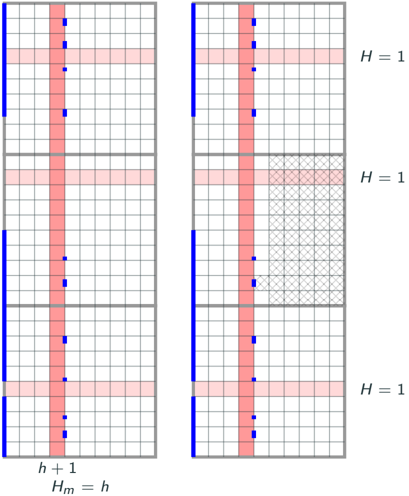
Use the control on v_k and get with high probability

$$|S'| \geq 400^{k+1/4}$$

after the defect.

Finally we use r_k and s_k to recover $|S''| \geq 400^{k+1}$ (w.h.p.).



Takeaways

Main takeaways:

- There is a “story-telling” in renormalization.
- Beautiful algebraic interplay between **environment** and **process**.
- There are many directions to go from here.

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"What is a sequence of i.i.d. Bernoulli random variables?"

Vladas Sidoravicius

Thank you!