Disordered systems and Hamilton-Jacobi equations.

I/ Motivations

Students $i \in \{1, \dots, N\}$.
two dorms ; assignment $\sigma \in \{\pm 1\}^N$.

Quality of interaction between i and j : J_{ij}.

Suppose (J_{ij}) are indep. standard Gaussian.

We want to maximize $\sigma \mapsto \sum J_{ij} \mathbb{1}_{\{\sigma_i = \sigma_j\}}$

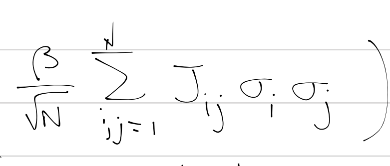
what is $\max_{\sigma \in \{\pm 1\}^N} \sum J_{ij} \sigma_i \sigma_j$ as $N \to +\infty$?

Frustrations

$$\mathbb{E} \frac{1}{N} \log \sum_{\sigma \in \{\pm 1\}^N} \exp\left(\frac{\beta}{\sqrt{N}} \sum_{i,j=1}^{N} J_{ij} \sigma_i \sigma_j \right)$$

$$\xrightarrow[N \to +\infty]{} \text{complicated formula.}$$

· Parisi '79 , Guerra '03 , Talagrand '06.



Rank-one matrix estimation

We observe a noisy version of a rank-one matrix.

Common thread . Hamilton-Jacobi equations.

Part I. - Curie-Weiss model.

I/ Definitions.

We want to study the proba. measure that, to each $\sigma \in \{\pm 1\}^N$, associates a weight prop. to

$$\exp\left(\frac{t}{N} \sum_{i,j=1}^{N} \sigma_i \sigma_j + h \sum_{i=1}^{N} \sigma_i \right)$$

$$\langle f(\sigma) \rangle_{t,h} = \frac{\sum_\sigma f(\sigma) \exp(\dots)}{\sum_\sigma \exp(\dots)} . \qquad t \geq 0, \; h \in \mathbb{R}.$$

$$F_N(t,h) = \frac{1}{N} \log \sum_\sigma \exp\left(\frac{t}{N} \sum \sigma_i \sigma_j + h \sum \sigma_i \right)$$

Moment generating function

$$\partial_h F_N = \frac{1}{N} \frac{\sum_\sigma (\sum \sigma_i) \exp(\dots)}{\sum_\sigma \exp(\dots)} = \frac{1}{N} \langle \sum \sigma_i \rangle$$

$$\partial_t F_N = \frac{1}{N} \left\langle \frac{1}{N} \sum_{i,j=1}^{N} \sigma_i \sigma_j \right\rangle = \left\langle \left(\frac{1}{N} \sum \sigma_i \right)^2 \right\rangle.$$

$$\partial_t F_N - (\partial_h F_N)^2 = \left\langle \left(\frac{1}{N} \sum \sigma_i \right)^2 \right\rangle - \left\langle \frac{1}{N} \sum \sigma_i \right\rangle^2.$$

$$\partial_h^2 F_N = \frac{1}{N} \left\langle \left(\sum \sigma_i \right)^2 \right\rangle - \frac{1}{N} \left(\langle \sum \sigma_i \rangle \right)^2$$

$$\partial_t F_N - (\partial_h F_N)^2 = \frac{1}{N} \partial_h^2 F_N$$

Moreover: $F_N(0,h) = \frac{1}{N} \log \sum_\sigma \exp\left(h \sum_i \sigma_i \right)$

$$= \frac{1}{N} \log \sum_\sigma \prod_{i=1}^{N} \exp(h \sigma_i)$$

$$= \frac{1}{N} \log \left(e^h + e^{-h} \right)^N$$

$$F_N(0,h) = F_1(0,h) = \psi(h)$$

II/ Interlude on Hamilton-Jacobi equations.

We need to think about what it means to be a solution of:

$$\partial_t f - (\partial_h f)^2 = 0. \qquad \boxed{\partial_t f - |\nabla f|^2 = 0}$$

Maybe we look for C^1 functions that satisfy the equation.

Pb :

$$\int \exp(\dots) \; dP^{\otimes N}(\sigma)$$

P proba. with bdd support on $\mathbb{R}$

$$\frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} J_{ij} \sigma_i \sigma_j \sim \mathcal{N}(0, N).$$

$$\max_{\sigma \in \{\pm 1\}^N} (E_\sigma) \sim \sqrt{N} \, c \sqrt{\log(2^N)}$$

$$\sim c N.$$

$\{1, \dots, N\}$.

$\sigma_i \in \{\pm 1\}$.

$\sigma_i \sigma_j$ " + noise".

Link between i & j with prob. p_N if $\sigma_i \sigma_j = 1$

q_N if $\sigma_i \sigma_j = -1$

$N p_N \to +\infty$ $\qquad$ $\boxed{p > q.}$