Simplicity and Complexity of Belief-Propagation

Elchanan Mossel[1]

[1]MIT

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A *Double* phase transition for large q

Theorem (Count Reconstruction, Robust Reconstruction (Mossel-Peres, Janson-Peres))

For all q and d-ary tree, $d\theta^2 = 1$ is the threshold for: census and robust reconstruction.

Theorem (Reconstruction for large q (Mossel 00))

If $d\theta > 1$ then for $q > q_\theta$ can distinguish the root better than random:

$$\lim_{h \to \infty} Var[\mathbb{E}[X_0 | X_{L_h}]] > 0$$

$\implies$ Non-linear estimators are superior.

Pf: Shows fractal nature of information.

Proof sketch

- For $q = \infty$, clearly threshold is $d\theta = 1$.
- For finite q,$d = 2$, fix θ such that $d\theta > 1$.
- Inference: Infer root color to be c if there is an ℓ-diluted binary subtree $T' \subset T$ with root at 0 and where all leaves have color c.
- Exercise 1: There exists an $\ell, \varepsilon > 0$ such that if the root is c, the probability that such a tree exists is at least ε.
- Exercise 2: For all $\varepsilon > 0$, if q is sufficiently large, and if the root is not c, the probability that there is an ℓ-diluted $2^\ell - 1$ tree with all the leaves of color $\neq c$ is at least $1 - \varepsilon/10$.
- Exercise 3: Prove that if $d\lambda \leq 1$, then the root and leaves are asymptotically independent.

More detailed Picture

- Sly 11: Defined magnetization $m_n = E[M_n]$ such that if m_n is small then:

$$m_{n+1} = d\theta^2 m_n + (1 + o(1))\frac{d(d-1)}{2}\frac{q(q-4)}{q-1}\theta^4 m_n^2.$$

- $\implies$ if $q \geq 5$, the KS bound is not tight.
- Also proved that if $q = 3$ and $d \geq d_{min}$ is large then KS bound is tight.
- M-01: For general Markov chains, can have $\lambda_2(M) = 0$, yet root and leaves are not independent.
- Exercise: Prove this for following chain on F_2^2.
  $M(x, y) = (r, r \oplus x)$ or $(r, r \oplus y)$ with probability $1/2$ each.
- More sophisticated examples in Mossel-Peres.

Two conjectures about inference

- Consider a model where different edges have different θ's.
- Let q so that for $\theta \in (\theta_R, \theta_{KS})$, $Var[\mathbb{E}[X_0|X_h]] \to \alpha > 0$.
- Conj 1: There is no estimator f such that $f(X_h)$ and X_0 have no negligible correlation for all models with $\theta(e) \in (\theta_R, \theta_{KS})$ for all edges.
- Conj 2: It is "impossible" to recover phylogenetic trees using $O(h)$ samples under the conditions above.
- Strong version of impossible would mean information theoretically. Weak version would mean computationally.

Part 3: Complexity of BP

Complexity of BP

- What is the complexity of BP?
- Low: Runs in linear time.
- But: Uses real numbers - it this necessary?
- But: Uses depth - is this necessary?
- Fractal picture suggests maybe depth is needed.

Understanding the Omnipresence

- What is everywhere and understand everything?
- “Omnipresence”.
- A: The deep-net on your smartphone that understands you.

Deep Inference?

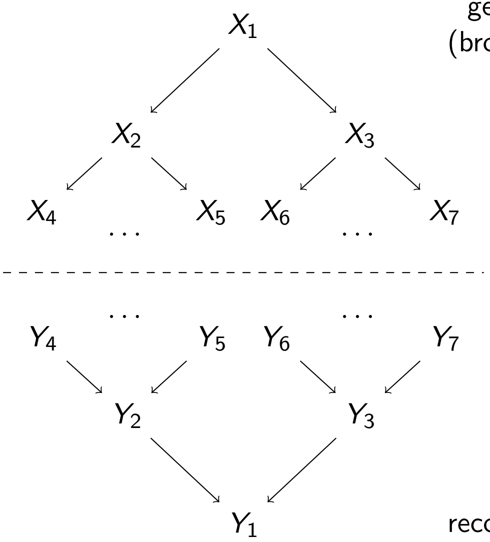
Mathematically, it is natural to ask if there are data generative process satisfying 3 natural criteria:

- 1. Realism: Reasonable data models.
- $\vee$
- 2. Reconstruction: Provable efficient algorithms to reverse engineer the generative process.
- $\vee$ (phylogenetic reconstruction).
- 3. Depth: Proof that depth is needed.
- ???
- 4. Also: why does BP use real numbers, when the generating process is discrete?

Precision in BP

- Q: What are the memory requirements for BP?
- Conjecture (EKPS-00): For $q = 2$, any recursive algorithm on the tree which uses at most B bits of memory per node can only distinguish the root value better then random if $\theta < \theta(B)$ where $d\theta(B)^2 > 1$.
- Thm:(Jain-Koehler-Liu-M-19): Conjecture is true: $\theta(B) - \theta = B^{-O(1)}$.

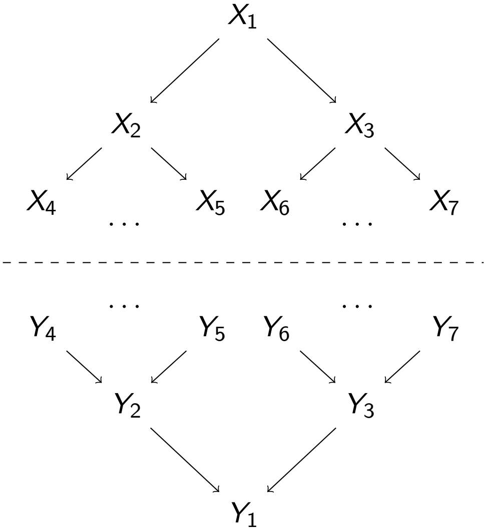
Problem Setup



generation tree (broadcast model)

reconstruction (message passing)

Problem Setup (cont.)



- Broadcast process on d-regular tree of height h.
- Each reconstruction $Y_i = f_i(Y_{2i}, Y_{2i+1})$ is an arbitrary $\log L$-bit string **(memory constraint)**.

$\mathbf{AC}^0$

- $\mathbf{AC}^0$:= class of bounded depth circuits with AND/OR (unbounded fan) and NOT gates.
- Thm: Moitra-M-Sandon-20:
- $\mathbf{AC}^0(X_h)$ cannot classify X_0 better than random.
- Is this trivial?
- Maybe not: Thm MMS-20: $\mathbf{AC}^0$ generates leaf distributions.

$\mathbf{TC}^0$

- $\mathbf{TC}^0$:= like $\mathbf{AC}^0$ but with Majority gates.
- "Bounded depth deep nets".
- Thm (MMS-20): When $q = 2$ and $0.9999 < \theta < 1$, there exists an algorithm A in $\mathbf{TC}^0$ such that $\lim_h P[A(X_h) = X_0] = \lim_h P[BP(X_h) = X_0]$.
- Conj: This is true for all θ when $q = 2$.
- So maybe we can classify optimally in $\mathbf{TC}^0$?
- Maybe bounded depth nets suffice?

$\mathbf{NC}^1$

- $\mathbf{NC}^1 :=$ class of $O(\log n)$ depth circuits with AND/OR (fan 2) and NOT gates.
- Known that $\mathbf{TC}^0 \subset \mathbf{NC}^1$. Open if they are the same.
- Thm (MMS-20): One can classify as well as BP in $\mathbf{NC}^1$.
- Thm (MMS-20): There is a broadcast process for which classifying better than random is $\mathbf{NC}^1$-complete.
- So, unless $\mathbf{TC}^0 = \mathbf{NC}^1$, $\log n$ depth is needed.

The KS bound and Circuit Complexity

- The threshold $2\theta^2 = 1$ is called the Kesten-Stigum threshold.
- Above this threshold it is known that one neuron can classify the root better than random (Kesten-Stigum-66).
- Below this threshold, one neuron cannot (M-Peres-04).
- Below this threshold, with enough i.i.d. noise on the leaves, BP becomes trivial (Janson-M-05).
- Related to "Replica Symmetry Breaking" in statistical physics models (Mezard-Montanari-06).
- Conjecture (MMS-20): For any broadcast process, below the KS bound and where BP classifies better than random, classification is $\mathbf{NC}^1$-complete.

Conclusion

- BP is simple:
  - Runs in linear time.
  - Above KS bound behaves like a Linear Algorithm.
- BP is complex:
  - Below KS bound, tend to be fractal.
  - Statistical/computation gaps.
  - Requires depth / precision.