Schramm-Loewner evolutions and imaginary geometry

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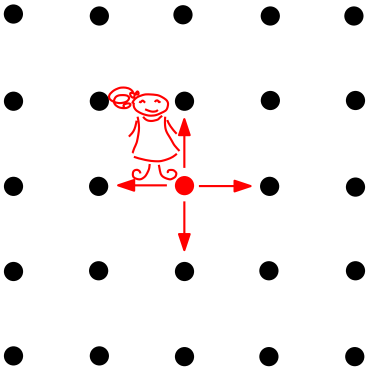
Outline

- Lecture 1: Definition and basic properties of SLE, examples
- Lecture 2: Basic properties of SLE
- Lecture 3: Imaginary geometry

References:

Conformally invariant processes in the plane by Lawler
Lectures on Schramm-Loewner evolution by Berestycki and Norris
Imaginary geometry I by Miller and Sheffield

Simple random walk



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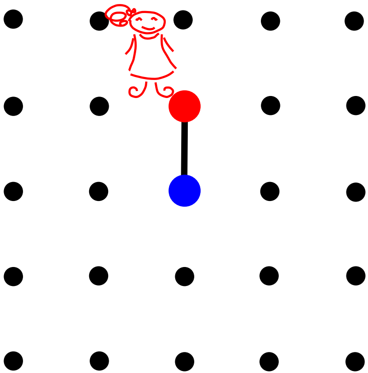

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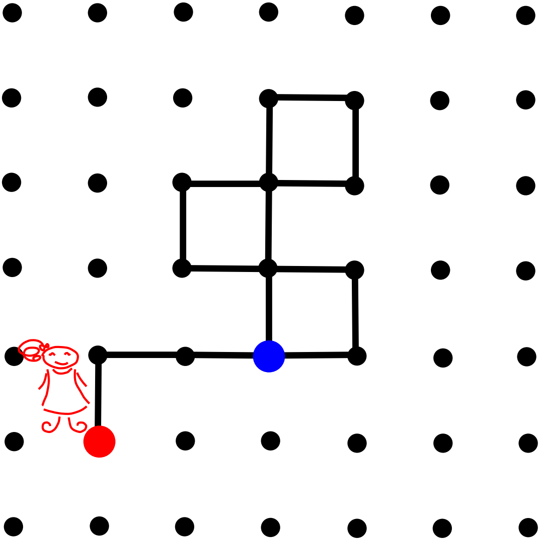

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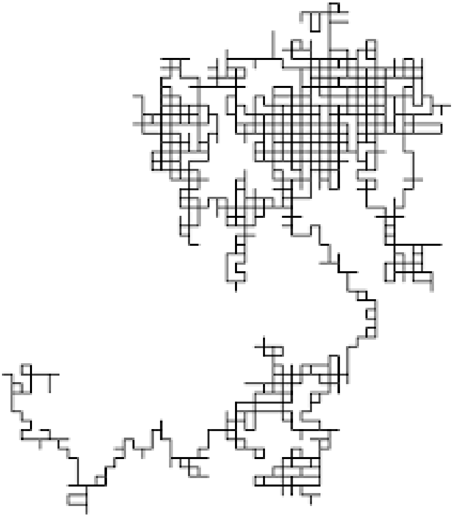

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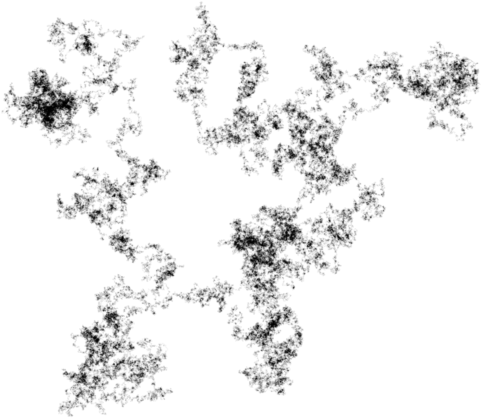
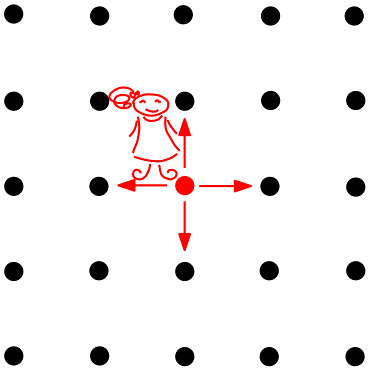


Donsker's theorem: Simple random walk converges to Brownian motion.

Loop-erased random walk (LERW)



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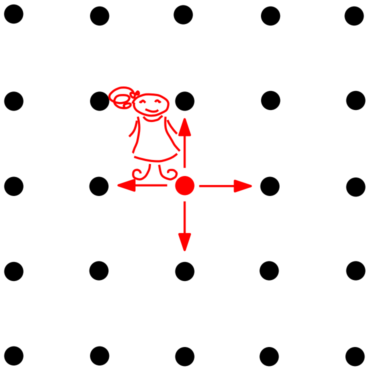

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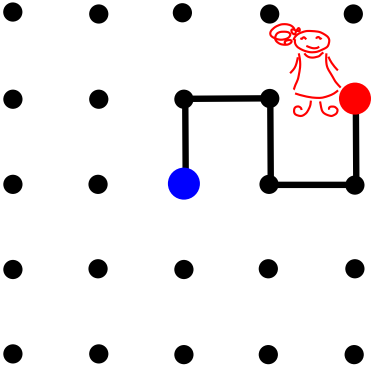

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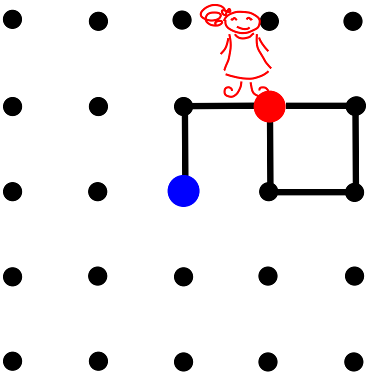

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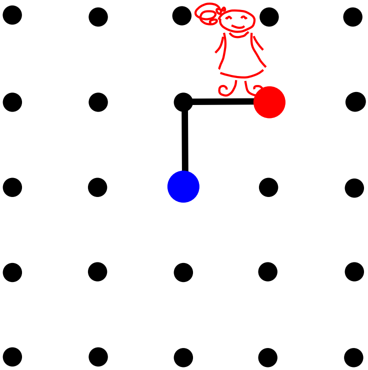

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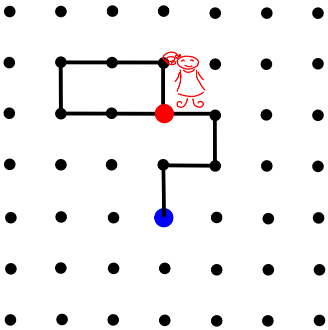

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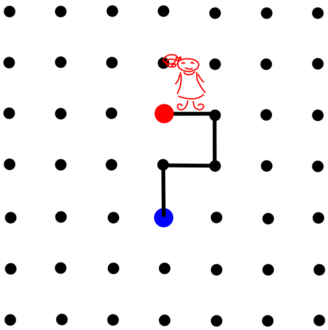
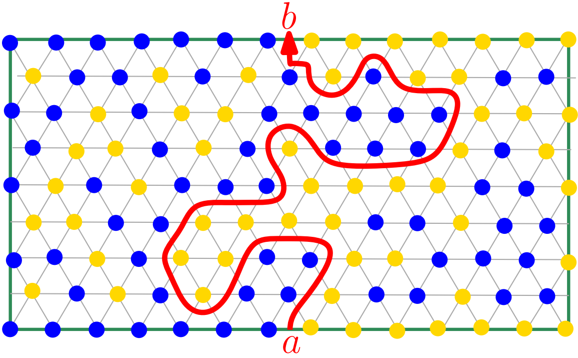


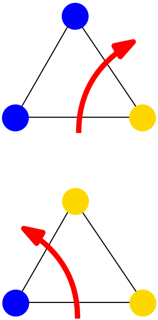
Loop-erased random walk (LERW)

- Lawler-Schramm-Werner'04: Loop-erased random walk $\Rightarrow$ SLE$_2$.

Illustration by P. Nolin

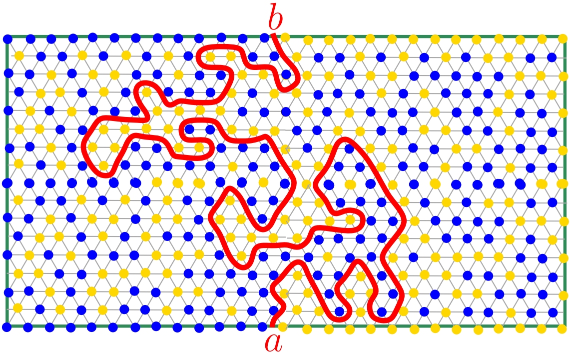
Critical percolation on the triangular lattice

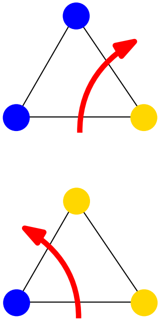




Smirnov'01: Critical percolation on the triangular lattice $\Rightarrow$ SLE_6

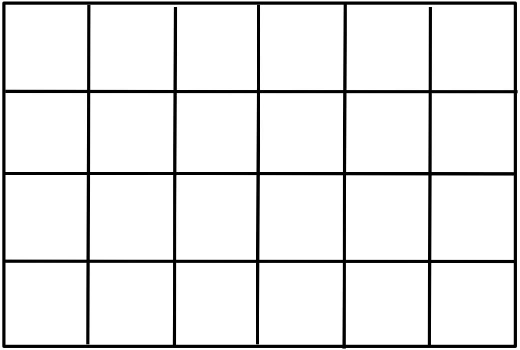
Critical percolation on the triangular lattice





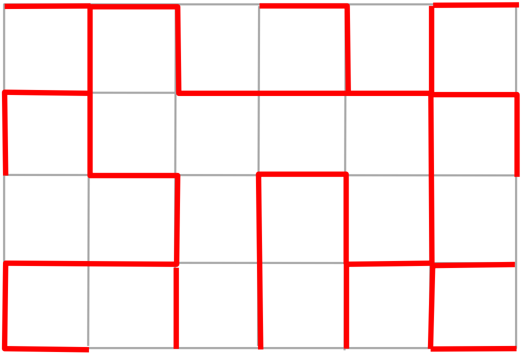
Smirnov'01: Critical percolation on the triangular lattice $\Rightarrow$ SLE_6

Uniform spanning tree (UST)



$\mathbb{Z}^2$ restricted to a box

Uniform spanning tree (UST)



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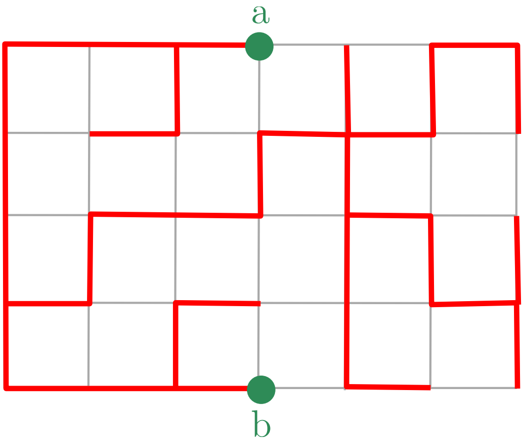
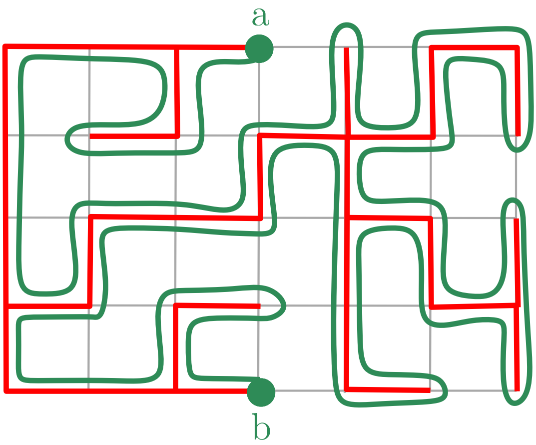


UST with wired *ab* boundary arc

Uniform spanning tree (UST)



Peano curve

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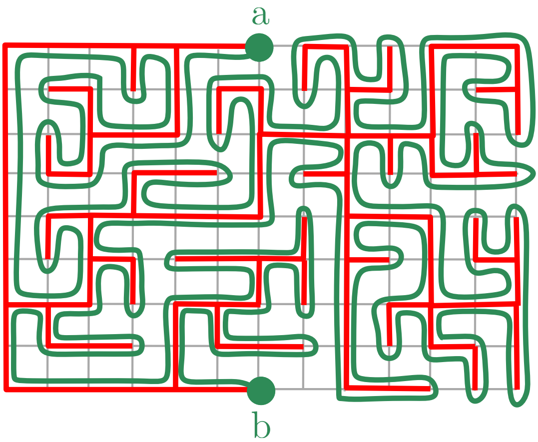
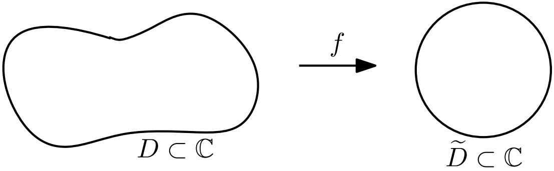


Peano curve

Lawler-Schramm-Werner'04: Peano curve of the UST $\Rightarrow$ SLE_8

Conformal maps



Definition (Conformal map)

f is conformal if f is bijective and f' exists.

$$f(z) = f_1(z_1, z_2) + if_2(z_1, z_2), \qquad z = z_1 + iz_2$$

Lemma (Cauchy-Riemann equations)

If f is conformal then

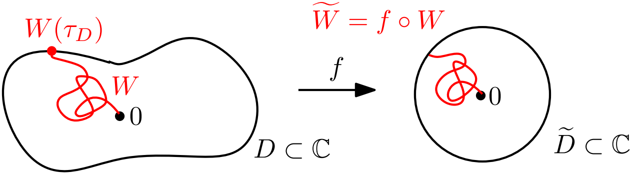
$$\partial_1 f_1 = \partial_2 f_2, \qquad \partial_2 f_1 = -\partial_1 f_2.$$

Conformal invariance of planar Brownian motion

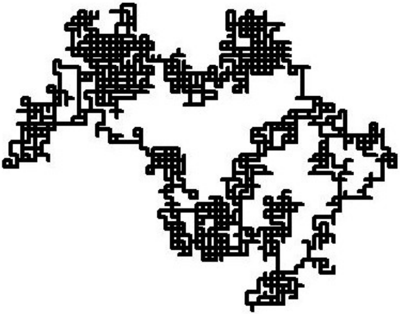
Theorem

- *Let W be a planar Brownian motion started from 0.*
- *Define $\tau_D := \inf\{t \geq 0 : W(t) \notin D\}$ for $D \subset \mathbb{C}$ a domain s.t. $0 \in D$.*
- *Let $f : D \to \widetilde{D}$ be a conformal map fixing the origin.*
- *Then $\widetilde{W} := f \circ W|_{[0,\tau_D]}$ has the law of a planar Brownian motion run until first leaving $\widetilde{D}$, modulo time reparametrization.*[a]

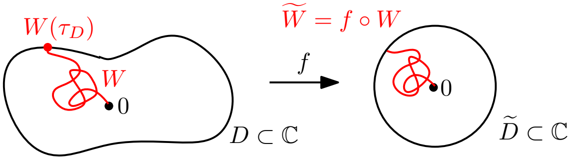
[a] We identify $w_1 : I_1 \to \mathbb{C}$ and $w_2 : I_2 \to \mathbb{C}$ (with $I_1, I_2 \subset \mathbb{R}$ intervals) if there is an increasing bijection $\phi : I_1 \to I_2$ such that $w_1 = w_2 \circ \phi$.



Conformal invariance of planar Brownian motion



Conformal invariance of Brownian motion: proof sketch



Theorem

$\widetilde{W} := f \circ W|_{[0,\tau_D]}$ has the law of a planar Brownian motion run until first leaving $\widetilde{D}$, modulo time reparametrization.

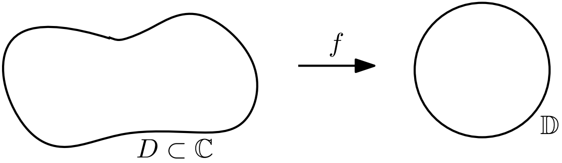
Write $\widetilde{W}(t) = \widetilde{W}_1(t) + i\widetilde{W}_2(t)$.

Exercise: Show that Itô's formula and the Cauchy-Riemann equations give

- $\widetilde{W}_1, \widetilde{W}_2$ are local martingales.
- $\langle \widetilde{W}_1 \rangle_t = \langle \widetilde{W}_2 \rangle_t$ and this function is a.s. strictly increasing in t.
- $\langle \widetilde{W}_1, \widetilde{W}_2 \rangle \equiv 0$.

These properties characterize a planar Brownian motion modulo time change (see e.g. Revuz-Yor).

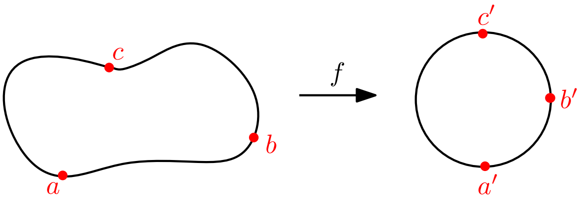
Riemann mapping theorem



Theorem (Riemann mapping theorem)

If D is a non-empty simply connected open proper subset of $\mathbb{C}$ then there exists a conformal map $f : D \to \mathbb{D}$.

Riemann mapping theorem



Theorem (Riemann mapping theorem)

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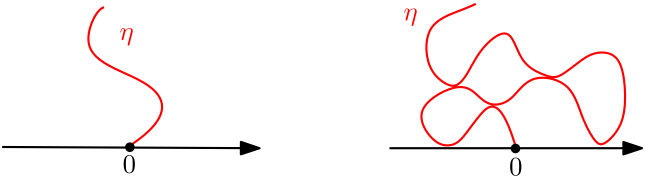
Three degrees of freedom.

Mapping out function

- $\eta : [0, \infty) \to \mathbb{H}$ curve in $\mathbb{H}$ from 0 to ∞.



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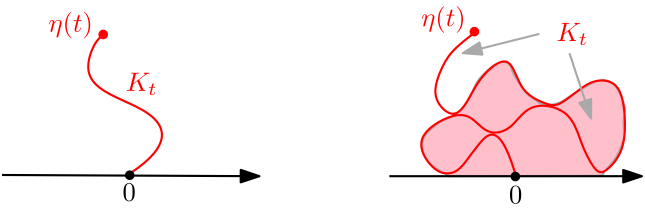

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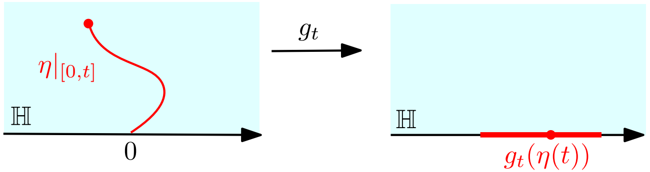

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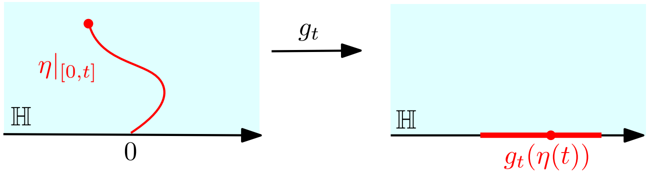
- $g_t(z) = a_1 z + a_0 + a_{-1} z^{-1} + \dots$ for $a_1, a_0, \dots \in \mathbb{R}$ near $z = \infty$
  - Show $\widetilde{g}_t(z) := -1/g_t(-z^{-1}) = \widetilde{a}_1 z + \widetilde{a}_2 z^2 + \dots$ by Schwarz reflection.



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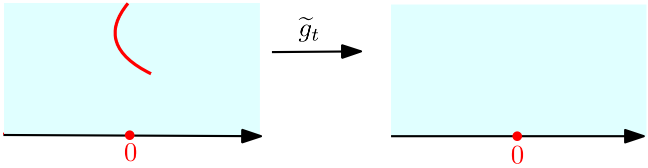

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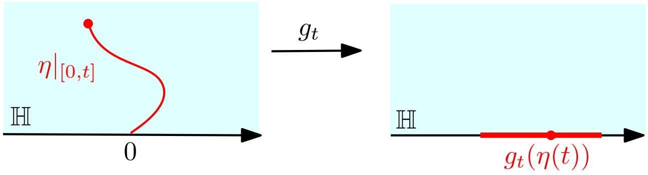
- Fix g_t by choosing $a_1 = 1$, $a_0 = 0$.



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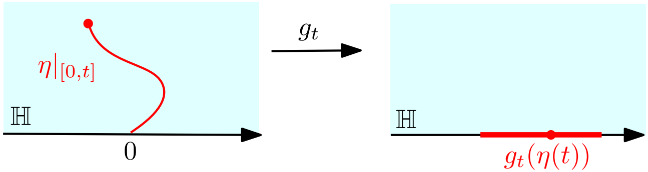
- Fix g_t by choosing $a_1 = 1$, $a_0 = 0$.
- g_t is the **mapping out function** of the **hull** K_t.



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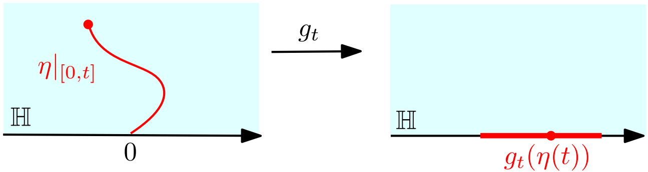
- Fix g_t by choosing $a_1 = 1, a_0 = 0$.
- g_t is the **mapping out function** of the **hull** K_t.
- Remark: Any compact $\mathbb{H}$-hull K (i.e., a bounded subset of $\mathbb{H}$ s.t. $\mathbb{H} \setminus K$ is open and simply connected) can be associated with a mapping out function $g : \mathbb{H} \setminus K \to \mathbb{H}$.

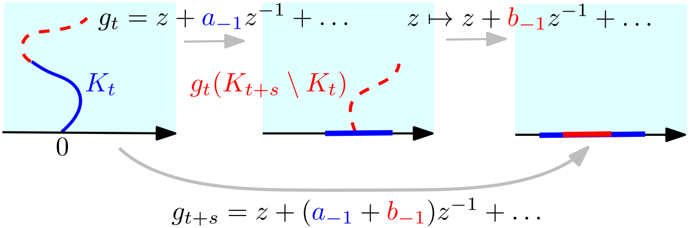


Half-plane capacity

Recall: $g_t(z) = z + a_{-1}z^{-1} + a_{-2}z^{-2} + \dots$
$\mathrm{hcap}(K_t) := a_{-1}$ is the "size" of K_t.

Lemma (additivity)

$\mathrm{hcap}(K_{t+s}) = \mathrm{hcap}(K_t) + \mathrm{hcap}(g_t(K_{t+s} \setminus K_t))$.



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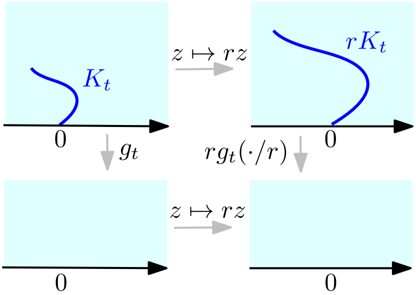
Lemma (scaling)

$\mathrm{hcap}(rK_t) = r^2\, \mathrm{hcap}(K_t)$



Observe that $\widetilde{g}_t(z) := rg_t(z/r)$ is the mapping out function of rK_t and that

$$\widetilde{g}_t(z) = z + r^2\, \mathrm{hcap}(K_t)z^{-1} + \dots$$

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Convention: Parametrize η such that $\text{hcap}(K_t) = 2t$.

Driving function and Loewner equation

η simple curve in $(\mathbb{H}, 0, \infty)$ parametrized by half-plane capacity.

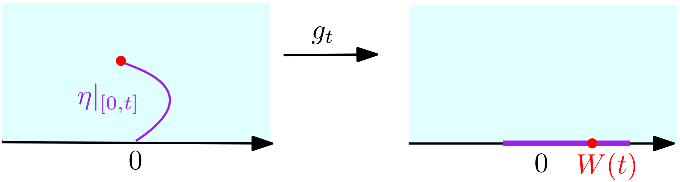
Definition (Driving function)

$W(t) := g_t(\eta(t))$

Proposition (Loewner equation)

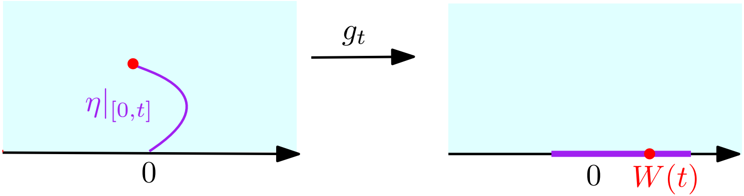
If $\tau_z = \inf\{t \geq 0 \, : \, z \in K_t\}$ *then*

$$\dot{g}_t(z) = \frac{2}{g_t(z) - W(t)} \quad \textit{for} \;\; t \in [0, \tau_z), \qquad g_0(z) = z \in \mathbb{H}.$$



Schramm's idea

- Key idea: study W instead of η.
- If η describes the conjectural scaling limit of certain discrete models, then W must be a multiple of a Brownian motion!



Definition of SLE_κ in $(\mathbb{H}, 0, \infty)$

- $\kappa \geq 0$ and $(B(t))_{t\geq 0}$ is a standard Brownian motion.
- Solve Loewner equation with driving function $W = \sqrt{\kappa} B$

$$\dot{g}_t(z) = \frac{2}{g_t(z) - W(t)}, \qquad \tau_z = \sup\{t \geq 0 \,:\, g_t(z) \text{ well-defined}\}.$$

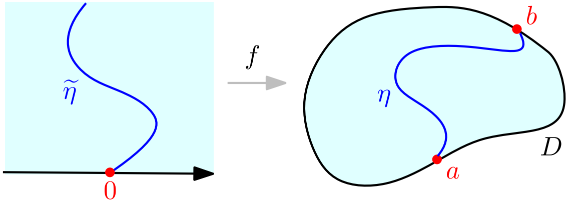
- Define $K_t := \{z \in \mathbb{H} \,:\, \tau_z \leq t\}$.
- Let η be the curve generating $(K_t)_{t\geq 0}$.
  - $K_t = \mathbb{H} \setminus \{\text{unbounded component of } \mathbb{H} \setminus \eta([0,t])\}$,
  - η is well-defined: Rohde-Schramm'05, Lawler-Schramm-Werner'04.

Definition (The Schramm-Loewner evolution in $(\mathbb{H}, 0, \infty)$)

η is an SLE_κ in $(\mathbb{H}, 0, \infty)$.

$$(B(t))_{t\geq 0} \rightarrow (g_t)_{t\geq 0} \rightarrow (K_t)_{t\geq 0} \rightarrow (\eta(t))_{t\geq 0}$$

Definition of SLE$_\kappa$ in (D, a, b)



Definition (The Schramm-Loewner evolution)

- Let $\widetilde{\eta}$ be an SLE$_\kappa$ in $(\mathbb{H}, 0, \infty)$.
- Then $\eta := f(\widetilde{\eta})$ is an SLE$_\kappa$ in (D, a, b).

- Note that f is not unique since $f \circ \phi_r$ also sends $(\mathbb{H}, 0, \infty)$ to (D, a, b) if $\phi_r(z) := rz$ for $r > 0$.
- SLE$_\kappa$ in (D, a, b) is still well-defined by scale invariance in law of SLE$_\kappa$ in $(\mathbb{H}, 0, \infty)$ (next slide).

Scale invariance in law of SLE$_\kappa$

Exercise (Scale invariance of SLE$_\kappa$)

- *Let* η *be an SLE*$_\kappa$ *in* $(\mathbb{H}, 0, \infty)$ *and let* $r > 0$.
- *Prove that* $t \mapsto r\eta(t/r^2)$ *has the law of an SLE*$_\kappa$ *in* $(\mathbb{H}, 0, \infty)$.

Scale invariance in law of SLE$_\kappa$

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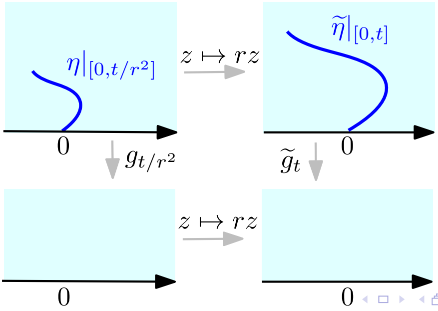
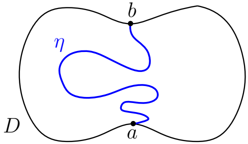
Hint: Let $\widetilde{\eta}(t) = r\eta(t/r^2)$ and argue that mapping out fcn $\widetilde{g}_t$ of $\widetilde{\eta}$ satisfy

$$\widetilde{g}_t(z) = rg_{t/r^2}(z/r), \qquad \dot{\widetilde{g}}_t(z) = \partial_t\left(rg_{t/r^2}(z/r)\right) = \frac{2}{\widetilde{g}_t(z) - rW(t/r^2)}.$$



Conformal invariance and domain Markov property

- Probability measure $\mu_{D,a,b}$ on curves η modulo time reparametrization in (D, a, b) for each simply connected domain $D \subset \mathbb{C}$, $a, b \in \partial D$.[1]



[1] Identify η and $\eta \circ \phi$ if $\phi : I_1 \to I_2$ cts and strictly increasing. ∂D Martin bdy of D.

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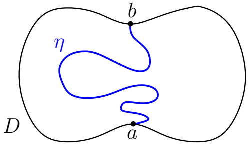

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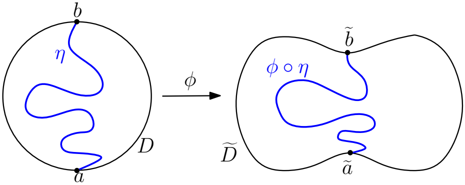

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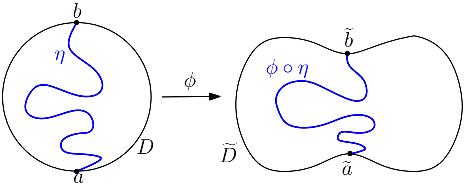
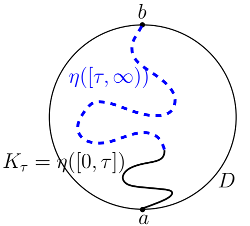


Conformal invariance



Domain Markov property

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Theorem (Schramm'00)

The following statements are equivalent:

- *$\mu_{D,a,b}$ satisfies (CI) and (DMP).*
- *There is a $\kappa \geq 0$ such that $\mu_{D,a,b}$ is the law of SLE$_\kappa$.*

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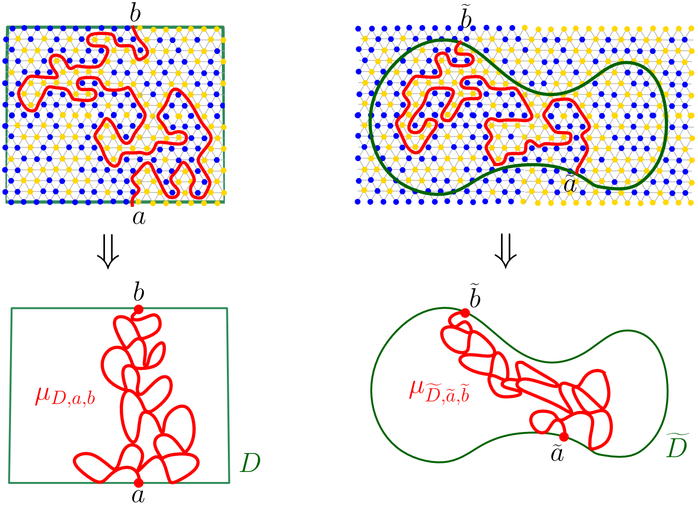
Conformal invariance of percolation



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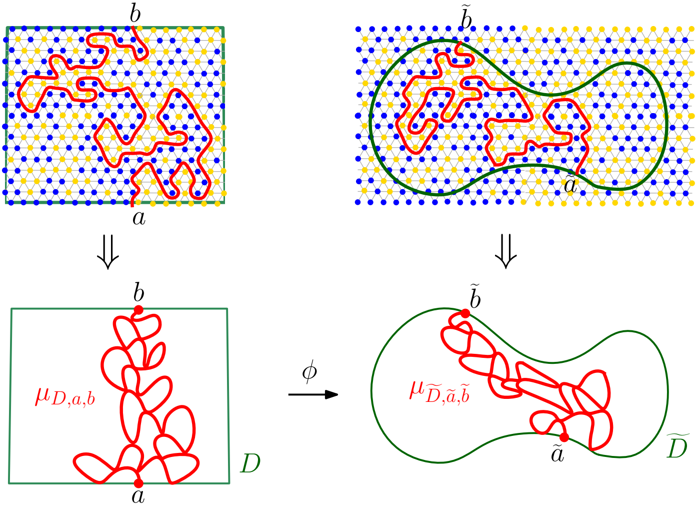


Conformal invariance: If $\eta \sim \mu_{D,a,b}$ then $\phi \circ \eta$ has law $\mu_{\widetilde{D},\tilde{a},\tilde{b}}$.

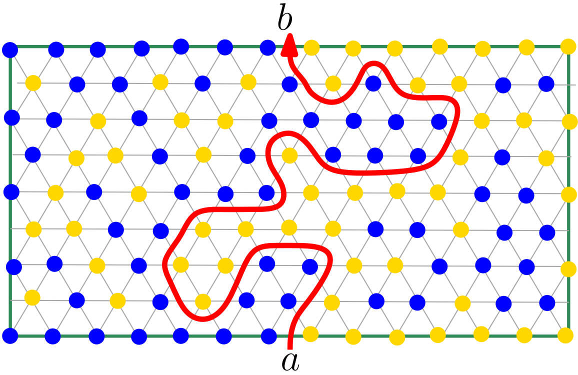
Outline

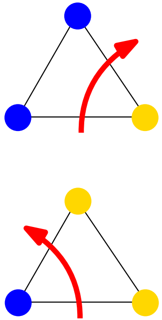
- Lecture 1: Definition and basic properties of SLE, examples
- **Lecture 2: Basic properties of SLE (today)**
- Lecture 3: Imaginary geometry

References:
Conformally invariant processes in the plane by Lawler
Lectures on Schramm-Loewner evolution by Berestycki and Norris
Imaginary geometry I by Miller and Sheffield

Key message today: The Loewner equation allows us to analyze SLE using stochastic calculus.

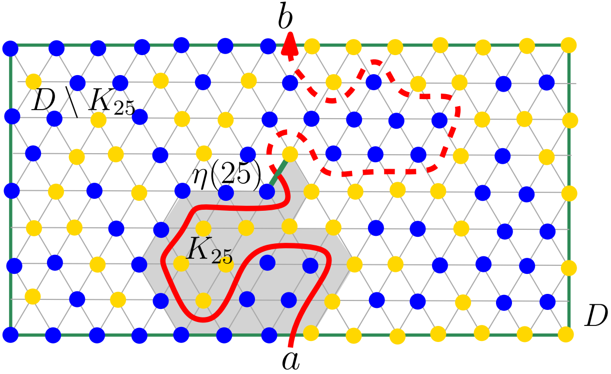
Domain Markov property of percolation

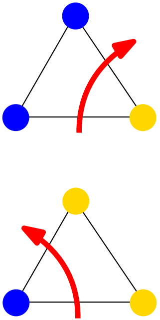




Domain Markov property of percolation

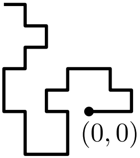
Conditioned on $\eta|_{[0,25]}$, the rest of the percolation interface has the law of a percolation interface in $(D \setminus K_{25}, \eta(25), b)$.





Domain Markov property of the self-avoiding walk

- Number of length n self-avoiding paths on $\mathbb{Z}^2$ from $(0,0)$: $\mu^{n(1+o(1))}$.



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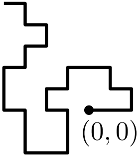

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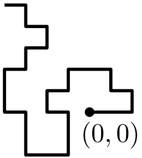
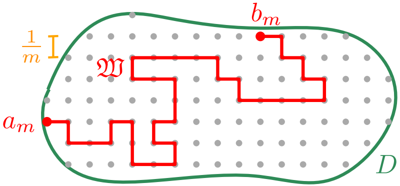
- **The self-avoiding walk (SAW)**: $\mathfrak{W}$ random path s.t. for w a self-avoiding path on discrete approximation (D_m, a_m, b_m) to (D, a, b),

$$\mathbb{P}[\mathfrak{W} = w] = c\mu^{-|w|},$$

where $|w|$ is the length of w and c is a renormalizing constant.





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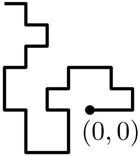
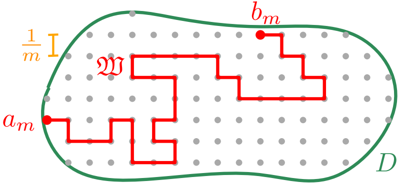
- Conjecture: $\mathfrak{W} \Rightarrow SLE_{8/3}$.





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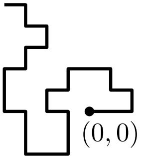
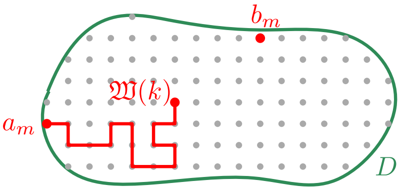
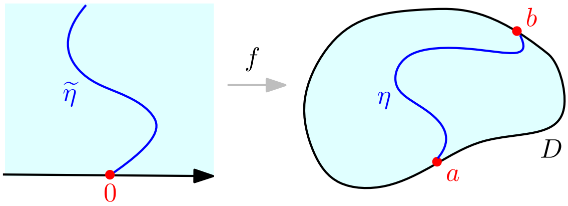
  where $|w|$ is the length of w and c is a renormalizing constant.
- Conjecture: $\mathfrak{W} \Rightarrow SLE_{8/3}$.
- Exercise: Given $\mathfrak{W}|_{[0,k]}$ the remaining path has the law of a SAW in $(D_m \setminus \mathfrak{W}([0,k]), \mathfrak{W}(k), b_m)$.





SLE satisfies (CI) and (DMP)

- (CI): follows from the definition of SLE_κ on general domains (D, a, b).



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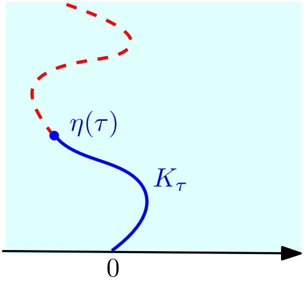


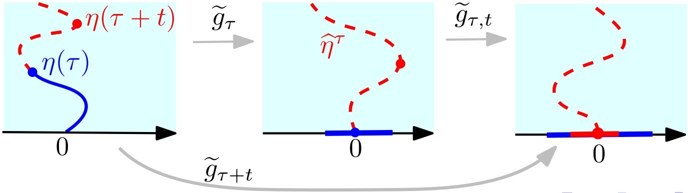
Want to prove: $\eta|_{[\tau,\infty)}$ has the law of an SLE_κ in $(\mathbb{H} \setminus K_\tau, \eta(\tau), \infty)$.

SLE satisfies (CI) and (DMP)

- (CI): follows from the definition of SLE_κ on general domains (D, a, b).
- (DMP): sufficient to verify for $(\mathbb{H}, 0, \infty)$ and parametrization by half-plane capacity.
  - Centered mapping out functions $\widetilde{g}_t(z) := g_t(z) - W(t)$ satisfy

$$d\widetilde{g}_t(z) = \frac{2}{\widetilde{g}_t(z)} - dW(t), \qquad \widetilde{g}_0(z) = z. \qquad \text{(CL)}$$

  - Exercise: Centered mapping out functions $(\widetilde{g}_{\tau,t})_{t\geq 0}$ of $\widehat{\eta}^\tau$ satisfy $\widetilde{g}_{\tau+t} = \widetilde{g}_{\tau,t} \circ \widetilde{g}_\tau$.
  - Exercise: Use previous exercise to argue that $(\widetilde{g}_{\tau,t})_{t\geq 0}$ satisfies (CL) w/driving function $(W(\tau + t) - W(\tau))_{t\geq 0} \stackrel{d}{=} (W(t))_{t\geq 0}$.
  - The last exercise implies that $\widehat{\eta}^\tau$ has the law of an SLE_κ in $(\mathbb{H}, 0, \infty)$.



(CI) and (DMP) imply that η is an SLE

- Suppose $(\mu_{D,a,b})_{D,a,b}$ satisfies (CI) and (DMP). Let $\eta \sim \mu_{\mathbb{H},0,\infty}$ be param. by half-plane capacity; let W denote the driving fcn of η.

(CI) and (DMP) imply that η is an SLE

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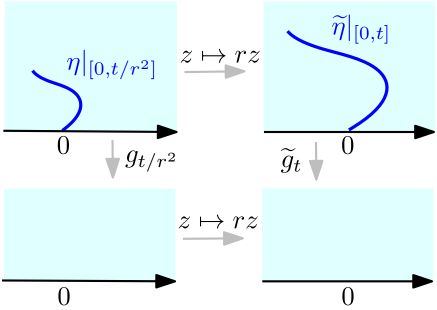
- (CI) $\Rightarrow$ scale invariance $\Rightarrow (W(t))_{t\geq 0} \overset{d}{=} (rW(t/r^2))_{t\geq 0}$.



Let $\widetilde{\eta}(t) := r\eta(t/r^2)$. Then $\eta \overset{d}{=} \widetilde{\eta}$. Mapping out fcn $(\widetilde{g}_t)_{t\geq 0}$ of $\widetilde{\eta}$ satisfy:

$$\widetilde{g}_t(z) = rg_{t/r^2}(z/r), \qquad \dot{\widetilde{g}}_t(z) = \partial_t\left(rg_{t/r^2}(z/r)\right) = \frac{2}{\widetilde{g}_t(z) - rW(t/r^2)}.$$

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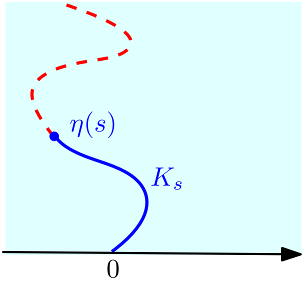
- (DMP)



(DMP): $\eta|_{[s,\infty)}$ has law $\mu_{\mathbb{H}\setminus K_s,\eta(s),\infty}$.

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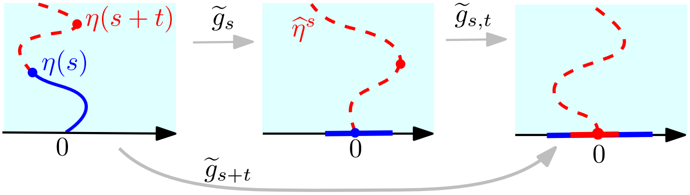
- (DMP) $\Rightarrow$ $(W(t))_{t\geq 0}$ has i.i.d. increments.
  - By (DMP), $\widehat{\eta}^s \stackrel{d}{=} \eta$ and $\widehat{\eta}^s$ is independent of $\eta|_{[0,s]}$.
  - The centered mapping out fcn $(\widetilde{g}_{s,t})_{t\geq 0}$ of $\widehat{\eta}^s$ satisfy the centered Loewner equation w/driving function $(W(s+t)-W(s))_{t\geq 0}$.
  - Combining the above, $(W(s+t)-W(s))_{t\geq 0} \stackrel{d}{=} (W(t))_{t\geq 0}$ and is independent of $W|_{[0,s]}$.



(CI) and (DMP) imply that η is an SLE

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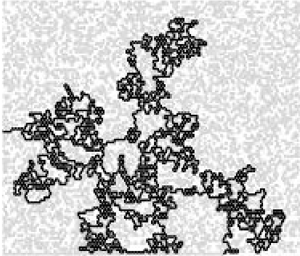
- (CI) $\Rightarrow$ scale invariance $\Rightarrow (W(t))_{t\geq 0} \stackrel{d}{=} (rW(t/r^2))_{t\geq 0}$.
- (DMP) $\Rightarrow (W(t))_{t\geq 0}$ has i.i.d. increments.
- (CI) + (DMP) $\Rightarrow W = \sqrt{\kappa}B$ for some $\kappa \geq 0$.

Phases of SLE

Rohde-Schramm'05: SLE_κ has the following phases:

- $\kappa \in [0,4]$: The curve is simple.
- $\kappa \in (4,8)$: The curve is self-intersecting and has zero Lebesgue measure.
- $\kappa \geq 8$: The curve fills space.

$\kappa \in [0,4]$



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Figures by P. Nolin, W. Werner, and J. Miller

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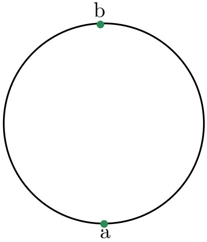

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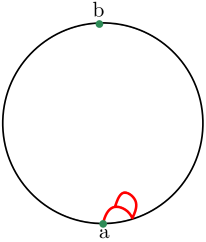

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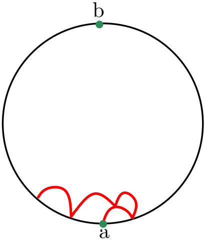

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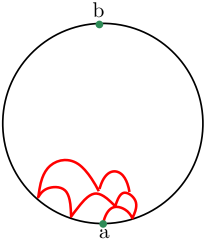

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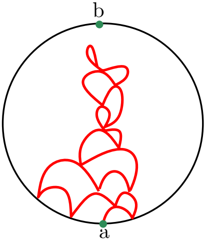

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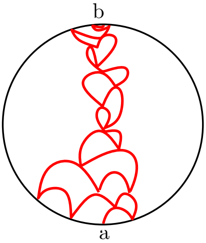

$\kappa \in (4,8)$

$\kappa \geq 8$

Figures by P. Nolin, W. Werner, and J. Miller

Rohde-Schramm'05: SLE_κ has the following phases:

- $\kappa \in [0,4]$: The curve is simple.
- $\kappa \in (4,8)$: The curve is self-intersecting and has zero Lebesgue measure.
- $\kappa \geq 8$: The curve fills space.

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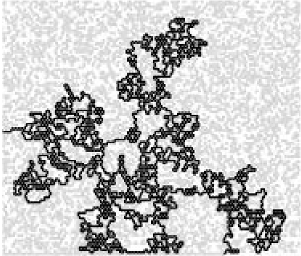

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Lemma

- *If* $\kappa \in [0,4]$ *then* η *is a.s. simple (i.e.,* $\eta(t_1) \neq \eta(t_2)$ *for* $t_1 \neq t_2$*).*
- *If* $\kappa > 4$ *then* η *is a.s. not simple.*

We will deduce the lemma from the following result, where

$$\tau_x = \inf\{t \geq 0 \,:\, x \in \overline{K}_t\} \text{ for } x > 0.$$

Lemma

- *If* $\kappa \in [0,4]$ *then* $\tau_x = \infty$ *a.s.*
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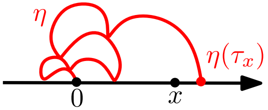

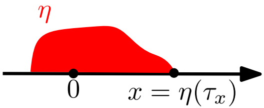

Phase transition at $\kappa = 4$

Recall $\tau_x = \inf\{t \geq 0 : x \in \overline{K}_t\}$ for $x > 0$.

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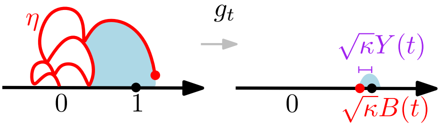

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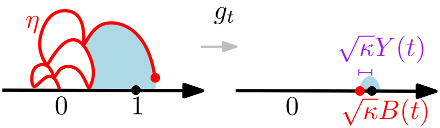

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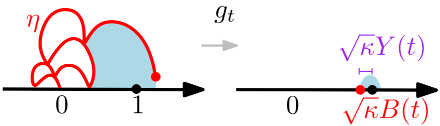
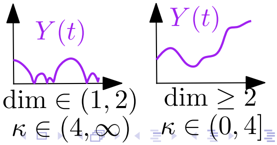
$dY(t) = \dfrac{2}{\kappa Y(t)}dt - dB(t)$, so $Y(t)$ is a $\left(\dfrac{4}{\kappa} + 1\right)$-dim. Bessel process.





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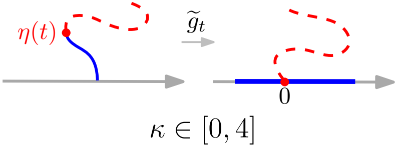

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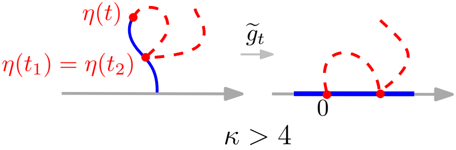

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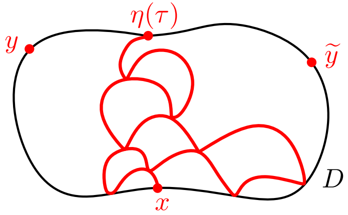
Locality of SLE_6

Proposition

- η SLE_6 in (D, x, y). Set $\tau := \inf\{t \geq 0 : \eta(t) \in \text{arc}(\widetilde{y}, y)\}$.
- Define $\widetilde{\eta}$ and $\widetilde{\tau}$ in the same way for $(D, x, \widetilde{y})$.
- Then $\eta|_{[0,\tau]} \overset{d}{=} \widetilde{\eta}|_{[0,\widetilde{\tau}]}$.



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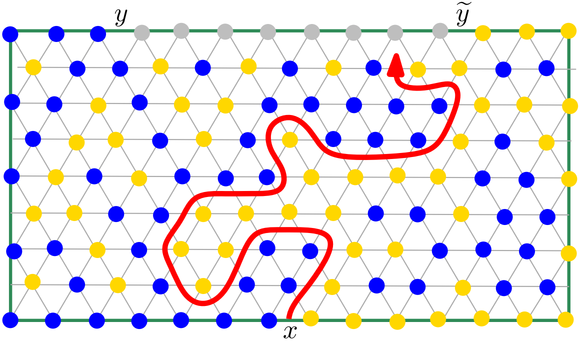

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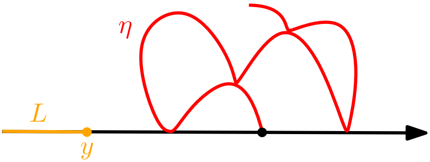
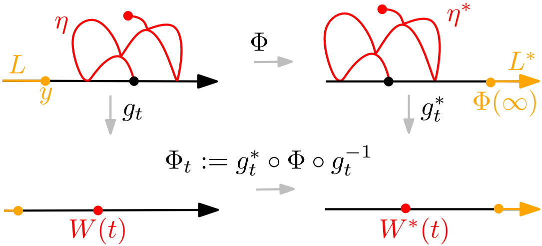


Want to prove: If η is an SLE_6 in $(\mathbb{H}, 0, \infty)$ then η has the law of an SLE_6 in $(\mathbb{H}, 0, y)$ until hitting L.

Locality of SLE$_6$: Proof sketch



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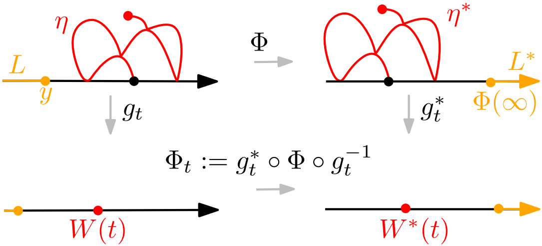

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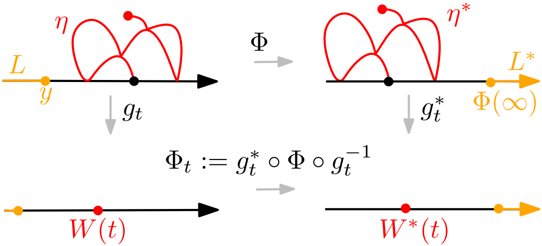

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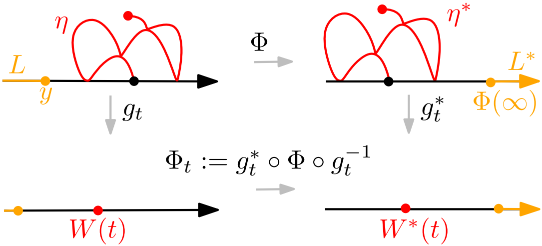

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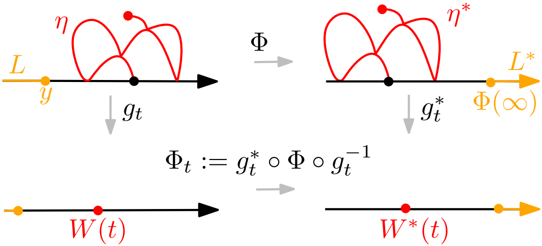

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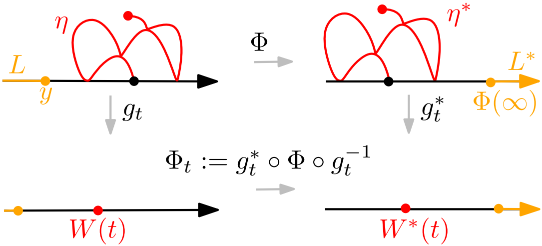

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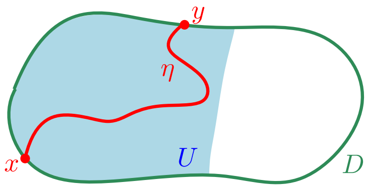
- Find dW^* by Itô's formula; prove and use $\dot{\Phi}_t(W(t)) = -3\Phi_t''(W(t))$.

Restriction property

Definition

- Let $\mu_{D,x,y}$ for $D \subset \mathbb{C}$ simply connected and $x, y \in \partial D$ be a family of probability measures on curves η in D from x to y.
- Let $\eta \sim \mu_{D,x,y}$ for some (D, x, y) and let $U \subset D$ be simply connected s.t. $x, y \in \partial U$.
- The measures $\mu_{D,x,y}$ satisfy the **restriction property** if η conditioned to stay in U has the law of a curve sampled from $\mu_{U,x,y}$.

For which $\kappa \geq 0$ does SLE$_\kappa$ satisfy the restriction property?



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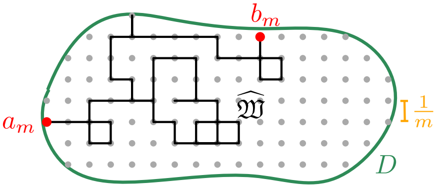

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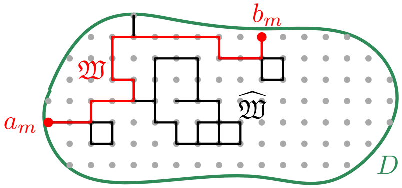

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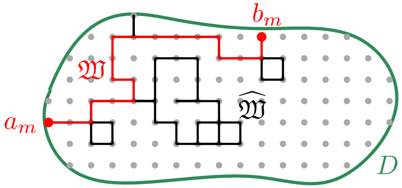

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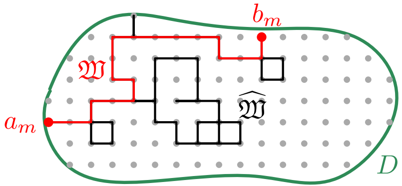
- "LERW in (D_m, a_m, b_m) conditioned to stay in U_m" $\neq$ "LERW in (U_m, a_m, b_m)", since the latter requires $\widehat{\mathfrak{W}} \subset U_m$ (not just $\mathfrak{W} \subset U_m$).



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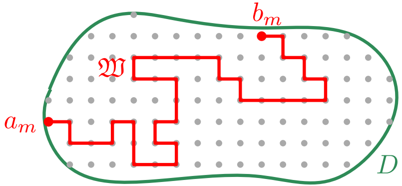
- Does the loop-erased random walk satisfy the restriction prop.? NO
- Does the self-avoiding walk satisfy the restriction property?

The **self-avoiding walk (SAW)** $\mathfrak{W}$ is s.t. for any fixed self-avoiding path w on discrete approximation (D_m, a_m, b_m) to (D, a, b),

$$\mathbb{P}[\mathfrak{W} = w] = c\mu^{-|w|},$$

where μ is the connective constant, $|w|$ is the length of w, and c is a renormalizing constant.



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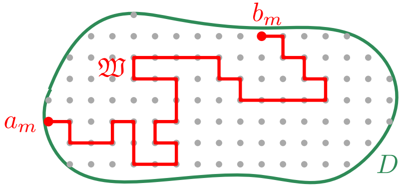
"SAW in (D_m, a_m, b_m) cond. to stay in U_m" $\overset{d}{=}$ "SAW in (U_m, a_m, b_m)"



Restriction property of $\mathrm{SLE}_{8/3}$

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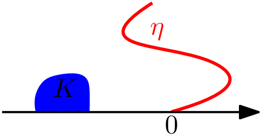

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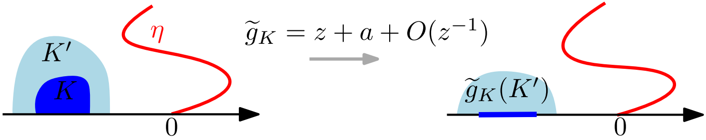
$$\mathbb{P}[\eta \cap K' = \emptyset \mid \eta \cap K = \emptyset] = \mathbb{P}[\eta \cap \widetilde{g}_K(K') = \emptyset], \qquad \text{(A)}$$

since RHS $= \mathbb{P}[\widehat{\eta} \cap K' = \emptyset]$ for $\widehat{\eta}$ an $\text{SLE}_{8/3}$ in $(\mathbb{H} \setminus K, 0, \infty)$.



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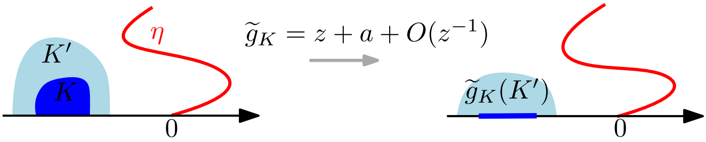

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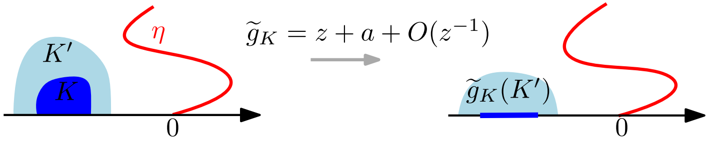
- This identity, Bayes' rule, and $\widetilde{g}_{K'} = \widetilde{g}_{\widetilde{g}_K(K')} \circ \widetilde{g}_K$ imply (A).



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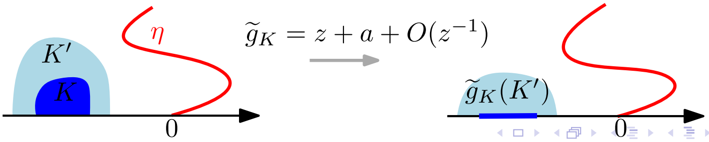
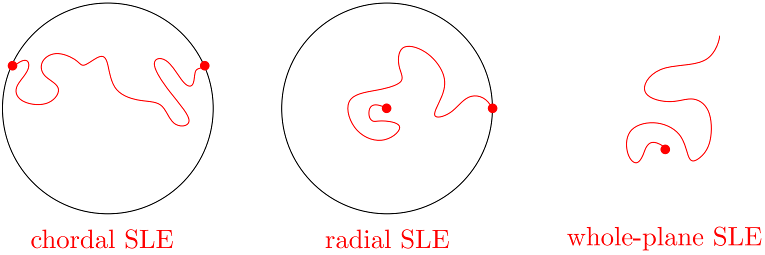
  since RHS $= \mathbb{P}[\widehat{\eta} \cap K' = \emptyset]$ for $\widehat{\eta}$ an $\mathrm{SLE}_{8/3}$ in $(\mathbb{H} \setminus K, 0, \infty)$.
- Key identity (proof omitted here): $\mathbb{P}[\eta \cap K = \emptyset] = \widetilde{g}'_K(0)^{5/8}$.
- This identity, Bayes' rule, and $\widetilde{g}_{K'} = \widetilde{g}_{\widetilde{g}_K(K')} \circ \widetilde{g}_K$ imply (A).
- Remark: Key identity with exponent $\alpha \geq 5/8$ represent other random sets satisfying conformal restriction.

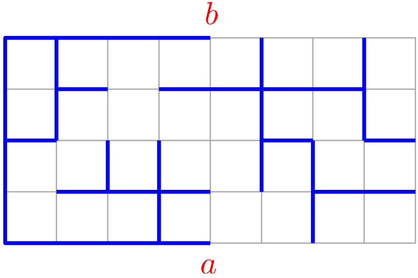


Chordal, radial, and whole-plane SLE



A few open questions

- Convergence of discrete models, e.g.
  - self-avoiding walk ($\kappa = 8/3$)
  - universality for percolation: $\mathbb{Z}^2$; Voronoi tesselation ($\kappa = 6$)
  - Fortuin-Kastelyn model ($\kappa \in (4, 8)$)
  - 6-vertex model ($\kappa = 12$, general κ)



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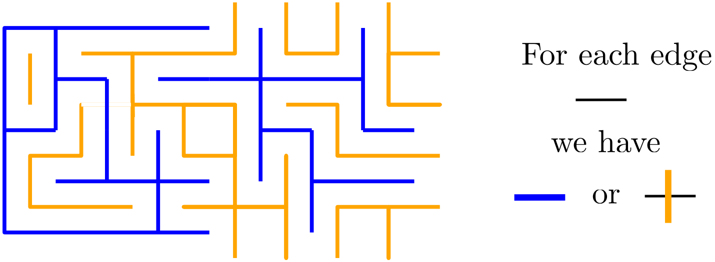

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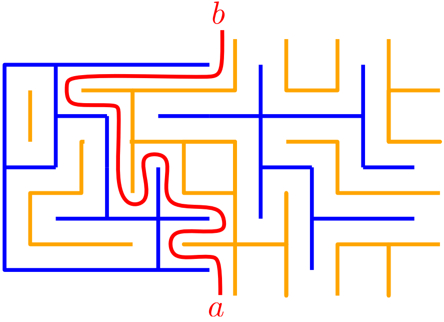

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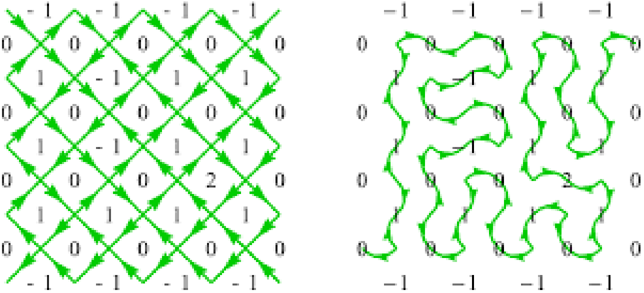

(a) 6-vertex configuration (b) Peano curve

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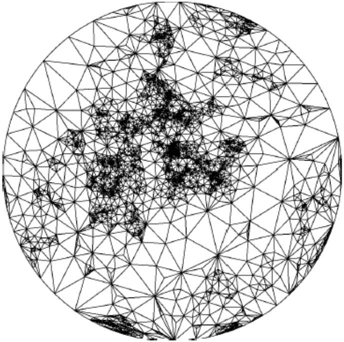

Random planar map; figure due to Gwynne-Miller-Sheffield

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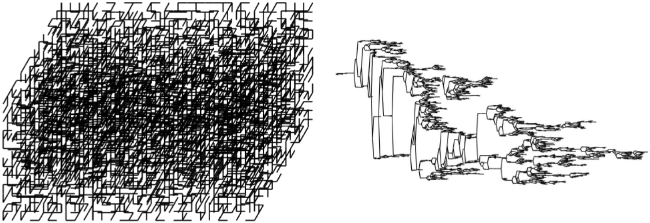

3d UST; figure by Angel–Croydon–Hernandez-Torres–Shiraishi

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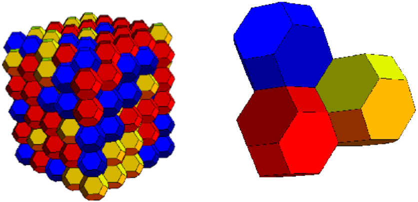

Figure by Sheffield-Yadin

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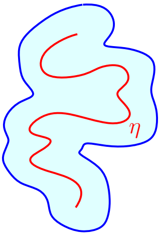
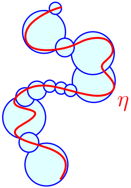
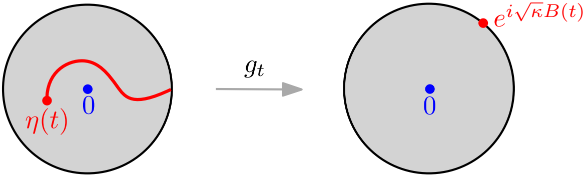
- Path properties of SLE, e.g.
  - Hausdorff measure of SLE





Thanks for attending!

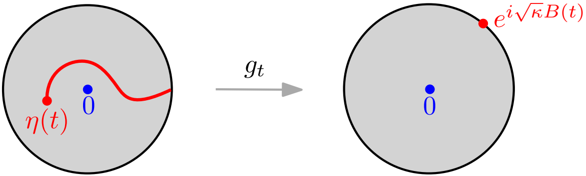
Radial SLE



- $g_t : \mathbb{D} \setminus K_t \to \mathbb{D}$ defined such that $g_t(0) = 0$ and $g_t'(0) > 0$.
- η parametrized such that $t = \log g_t'(0)$.
- Radial Loewner equation, where B is a standard Brownian motion

$$\dot{g}_t(z) = g_t(z) \frac{e^{i\sqrt{\kappa}B(t)} + g_t(z)}{e^{i\sqrt{\kappa}B(t)} - g_t(z)}, \qquad g_0(z) = z.$$

Radial SLE



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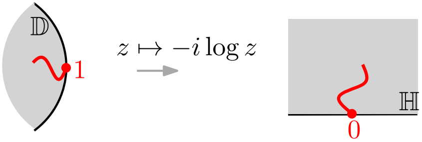

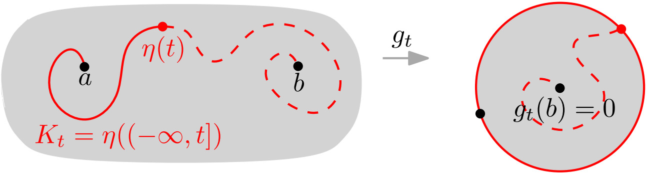

Conditioned on $\eta|_{(-\infty,t]}$, the remainder $\eta|_{(t,\infty)}$ of the curve has the law of radial SLE_κ in $(\mathbb{C} \setminus K_t, \eta(t), b)$.