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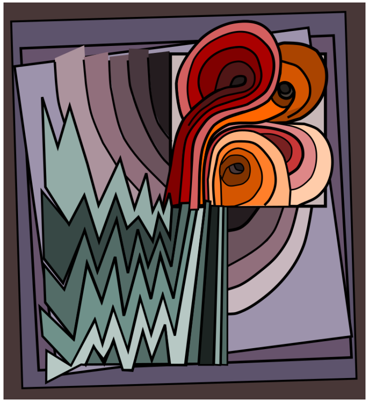
Facets of stochastic quantisation

lecture 1



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outline

- euclidean quantum fields
- what is stochastic quantisation?
- varieties of stochastic quantisation
- infinite volume limit ($L \to \infty$)
- renormalization and small scale limit ($\varepsilon \to 0$)
- properties of stochastically quantised measures
- elliptic stochastic quantisation (?) & supersymmetry

reference material

https://www.iam.uni-bonn.de/abteilung-gubinelli/sq-lectures-milan-ws2021

Euclidean quantum fields (EQFs)

are particular class of probability measures on $\mathscr{S}'(\mathbb{R}^d)$:

$$\int_{\mathscr{S}'(\mathbb{R}^d)} O(\varphi)\nu(\mathrm{d}\varphi) = \frac{1}{Z}\int_{\mathscr{S}'(\mathbb{R}^d)} O(\varphi)e^{-S(\varphi)}\mathrm{d}\varphi,$$

$$S(\varphi) = \int_{\mathbb{R}^d} \frac{1}{2}|\nabla\varphi(x)|^2 + \frac{1}{2}m^2|\varphi(x)|^2 + V(\varphi(x))\mathrm{d}x$$

for some non-linear function $V: \mathbb{R} \to \mathbb{R}_{\geqslant 0}$, e.g. a polynomial bounded below, exponentials, trig funcs.

Introduced in the '70-'80 as a tool to constructs models of (bosonic) quantum field theories in the sense of Wightman via the reconstruction theorem of Osterwalder–Schrader.

ill-defined representation:

- **large scale (IR) problems:** the integral in $S(\varphi)$ extends over all the space, sample paths not expected to decay at infinity in any way.
- **small scale (UV) problems:** sample paths are not expected to be function, but only distributions, the quantity $V(\varphi(x))$ does not make sense.

▹ Construct rigorously QM models which are compatible with special relativity, (finite speed of signals and Poincaré covariance of Minkowski space $\mathbb{R}^{n+1}$).

▹ Quantum field theory (QM with ∞ many degrees of freedom)

▹ Wightman axioms ('60-'70): Hilbert space, representation of the Poincaré group, fields operators (to construct local observables).

▹ Constructive QFT program: Hard to find models of such axioms. Examples in $\mathbb{R}^{1+1}$ were found in the $'60$.

▹ Euclidean rotation: $t \to it = x_0$ (imaginary time). $\mathbb{R}^{n+1} \to \mathbb{R}^d$ Minkowski → Euclidean

▹ Osterwalder–Schrader theorem : gives precise condition to perform the passage to/from Euclidean space (OS axioms for Euclidean correlation function).

▹ Surprise: in some cases the Euclidean theory is a probability measure on $\mathscr{S}'(\mathbb{R}^d)$.

▹ High point of CQFT: construction of Φ^4_3 (Euclidean version of a scalar field in $\mathbb{R}^{2+1}$ Minkowski space).

EQFs – for us

An EQFT is a prob. measure μ on $\mathscr{S}'(\mathbb{R}^d)$ such that the following holds (OS axioms)

1. **Regularity**: $\int_{\mathscr{S}'(\mathbb{R}^d)} e^{\alpha\|\varphi\|_s}\mu(\mathrm{d}\varphi)<\infty$ where $\|\varphi\|_s$ is some norm on $\mathscr{S}'(\mathbb{R}^d)$ and $\alpha>0$.
2. **Euclidean covariance**: The Euclidean group G (rotation+translation) acts on $\mathscr{S}'(\mathbb{R}^d)$ and the measure μ is invariant under this action. Example:

$$\int_{\mathscr{S}'(\mathbb{R}^d)} \varphi(f(\cdot+h))\mu(\mathrm{d}\varphi)=\int_{\mathscr{S}'(\mathbb{R}^d)} \varphi(f(\cdot))\mu(\mathrm{d}\varphi), \qquad f\in\mathscr{S}(\mathbb{R}^d).$$

3. **Reflection positivity**: Let $\theta(x_1,\ldots x_d)=(-x_1,x_2,\ldots,x_d)\in\mathbb{R}^d$, then for any bounded measurable $F\colon\mathscr{S}'(\mathbb{R}_{>0}\times\mathbb{R}^{d-1})\to\mathbb{C}$ we have

$$\int \overline{F(\theta\varphi)}F(\varphi)\mu(\mathrm{d}\varphi)\geqslant 0.$$

Example: for $x_1>0$, $\varphi(x_1,x_2,\ldots x_d)=\varphi(\delta_{(x_1,x_2,\ldots x_d)})$, $(x_1,x_2,\ldots x_d)\in\mathbb{R}^d$

$$\int \varphi(-x_1,x_2,\ldots,x_d)\varphi(x_1,x_2,\ldots x_d)\mu(\mathrm{d}\varphi)\geqslant 0, \qquad \int \varphi(y)\varphi(y')\overline{\varphi(\theta y)}\,\overline{\varphi(\theta y')}\mu(\mathrm{d}\varphi)\geqslant 0.$$

Simplest example of EQFT. We take a Gaussian measure μ on $\mathscr{S}'(\mathbb{R}^d)$ with covariance

$$\int \varphi(x)\varphi(y)\mu(\mathrm{d}\varphi) = G(x-y) = \int_{\mathbb{R}^d} \frac{e^{ik(x-y)}}{m^2 + |k|^2} \frac{\mathrm{d}k}{(2\pi)^d} = (m^2 - \Delta)^{-1}(x-y), \qquad x, y \in \mathbb{R}^d$$

and zero mean. Reflection positive, Eucl. covariant and regular. This is the GFF with mass $m > 0$. This measure can be used to construct a QFT in Minkowski space but unfortunately this theory is free, i.e. there is no interaction.

Other gaussian measures which are reflection positive, Eucl. covariant and regular can be constructed by positive linear combinations of Gaussians, i.e. by taking

$$\int \varphi(x)\varphi(y)\mu(\mathrm{d}\varphi) = \int_{\mathbb{R}_+} \lambda(\mathrm{d}r) \int_{\mathbb{R}^d} \frac{e^{ik(x-y)}}{r + |k|^2} \frac{\mathrm{d}k}{(2\pi)^d}.$$

These are the only known RP Gaussian measures.

Note that $G(0) = +\infty$ if $d \geqslant 2$, this implies that the GFF is not a function.

add interaction

Can we construct a non-Gaussian EQFT?

The heuristic idea is to try to maintain the “Markovianity” of the GFF μ. Heuristically we want something like

$$\nu(\mathrm{d}\varphi) = \frac{e^{\int_\Lambda V(\varphi(x))\mathrm{d}x}}{Z}\mu(\mathrm{d}\varphi),$$

with $\Lambda = \Lambda_+ \cup \theta\Lambda_+$ and $V\colon \mathbb{R} \to \mathbb{R}$ so that

$$\int_\Lambda V(\varphi(x))\mathrm{d}x = \int_{\Lambda_+} V(\varphi(x))\mathrm{d}x + \int_{\Lambda_+} V((\theta\varphi)(x))\mathrm{d}x$$

since it will be RP:

$$\int \overline{F(\theta\varphi)}F(\varphi)\nu(\mathrm{d}\varphi) = \int \frac{\overline{F(\theta\varphi)e^{\int_{\Lambda_+} V(\theta\varphi(x))\mathrm{d}x}}F(\varphi)e^{\int_{\Lambda_+} V(\varphi(x))\mathrm{d}x}}{Z}\mu(\mathrm{d}\varphi) \geqslant 0.$$

Unfortunately even if we can make sense of it this measure will not be translation invariant, ideally we would like to have $\Lambda = \mathbb{R}^d$.

❶ go on a lattice: $\mathbb{R}^d \to \mathbb{Z}^d_\varepsilon = (\varepsilon\mathbb{Z})^d$ with spacing $\varepsilon > 0$ and make it periodic $\mathbb{Z}^d_\varepsilon \to \mathbb{Z}^d_{\varepsilon,L} = (\mathbb{Z}_\varepsilon / 2\pi L \mathbb{N})^d$.

$$\int F(\varphi)\nu^{\varepsilon,L}(\mathrm{d}\varphi) = \frac{1}{Z_{\varepsilon,L}} \int_{\mathbb{R}^{\mathbb{Z}^d_{\varepsilon,L}}} F(\varphi) e^{\overbrace{-\frac{1}{2}\sum_{x \in \mathbb{Z}^d_{\varepsilon,L}} |\nabla_\varepsilon \varphi(x)|^2 + m^2 \varphi(x)^2 + V_\varepsilon(\varphi(x))}^{S_\varepsilon(\varphi)}} \mathrm{d}\varphi$$

RP (on the torus) + translation invariant (on the lattice). Lost rotations.

❷ Given $\varphi \in \mathbb{R}^{\mathbb{Z}^d_{\varepsilon,L}}$ and $f \in \mathscr{S}(\mathbb{R}^d)$ then we can define

$$\varphi(f) = \sum_{x \in \mathbb{Z}^d_{\varepsilon,L}} \varphi(x) f(x)$$

which allows to look at $\nu^{\varepsilon,L}$ as a measure on $\mathscr{S}'(\mathbb{R}^d)$.

ε is an UV regularisation and L the IR one.

❸ choose V_ε appropriately so that $v^{\varepsilon,L} \to v$ to some limit as $\varepsilon \to 0$ and $L \to \infty$. We take V_ε polynomial bounded below (otherwise integrab. problems). $d = 2, 3$.

$$V_\varepsilon(\xi) = \lambda(\xi^4 - a_\varepsilon \xi^2)$$

The limit measure will depend on $\lambda > 0$ and on $(a_\varepsilon)_\varepsilon$ which has to be s.t. $a_\varepsilon \to +\infty$ as $\varepsilon \to 0$. It is called the Φ^4_d measure.

▹ for $d = 2$ other choices are possible:

$$V_\varepsilon(\xi) = \lambda \xi^{2l} + \sum_{k=0}^{2l-1} a_{k,\varepsilon} \xi^k, \quad V_\varepsilon(\xi) = a_\varepsilon \cos(\beta\xi)$$

$$V_\varepsilon(\xi) = a_\varepsilon \cosh(\beta\xi), \qquad V_\varepsilon(\xi) = a_\varepsilon \exp(\beta\xi)$$

▹ for $d = 3$ “only” 4th order (6th order is critical).

▹ for $d = 4$ all the possible limits are Gaussian (see recent work of Aizenmann-Duminil Copin, arXiv:1912.07973)

We are interested in limits of quantities like

$$\lim_{\varepsilon \to 0, L \to \infty} \int \varphi(f_1) \cdots \varphi(f_n) \nu^{\varepsilon, L}(\mathrm{d}\varphi) = \int \varphi(f_1) \cdots \varphi(f_n) \nu(\mathrm{d}\varphi)$$

for arbitrary test functions $f_1, \ldots, f_n \in \mathscr{S}(\mathbb{R}^d)$. For $d = 2, 3$ problem solved in $'70 - '80$ by Glimm, Jaffe, ...

Parisi-Wu, Neson ('84): introduce a stochastic differential equation (SDE) which has ν as invariant measure. For clarity we work with $\nu^{\varepsilon, L}$. The SDE is a Langevin equation of the form

$$\frac{\mathrm{d}\Phi(t, x)}{\mathrm{d}t} = -\nabla_\varphi S_\varepsilon(\Phi(t, x)) + 2^{1/2} \xi(t, x), \qquad x \in \Lambda_{\varepsilon, L} = \mathbb{Z}^d_{\varepsilon, L}, t \geqslant 0$$

Here $\xi(t, x)$ is a space-time white noise.

If $\mathrm{Law}(\Phi(t = 0)) = \nu^{\varepsilon, L}$ then $\mathrm{Law}(\Phi(t)) = \nu^{\varepsilon, L}$ for all $t \geqslant 0$.

Usually more is true, for any $\varphi(0)$ on has that $\mathrm{Law}(\varphi(t)) \to \nu^{\varepsilon,L}$ as $t \to \infty$

$$\frac{\mathrm{d}\Phi(t,x)}{\mathrm{d}t} = -\nabla_\varphi S_\varepsilon(\Phi(t,x)) + 2^{1/2}\xi(t,x)$$

$$\frac{\mathrm{d}\Phi_{\varepsilon,L}(t,x)}{\mathrm{d}t} = -(m^2 - \Delta_\varepsilon)\Phi_{\varepsilon,L}(t,x) - V'_\varepsilon(\Phi_{\varepsilon,L}(t,x)) + 2^{1/2}\xi(t,x)$$

A (discrete) parabolic SPDE.

$$\nu^{\varepsilon,L} \sim \Phi_{\varepsilon,L}(t) = G_{\varepsilon,L}(\Phi_{\varepsilon,L}(0),\xi).$$

If we can take the $t \to \infty$ limit we expected that

$$\nu^{\varepsilon,L} \sim \Phi_{\varepsilon,L}(\infty) = \hat{G}_{\varepsilon,L}(\xi).$$

The idea now is to use this equation to take the limit $\varepsilon \to 0$. $L \to \infty$.

Why is this a good idea?

stochastic quantisation is a **stochastic analysis** of EQFs

The dynamics construct for you a map $\hat{G}_{\varepsilon,L}$ which transform a gaussian measure into $\nu^{\varepsilon,L}$

In particular this map passes to the limit as $\varepsilon \to 0$ and $L \to \infty$ and give a SPDE in the limit

$$\frac{\mathrm{d}\Phi(t,x)}{\mathrm{d}t} = -(m^2 - \Delta)\Phi(t,x) - '' V'(\Phi(t,x))'' + 2^{1/2}\xi(t,x).$$

The goal of these lectures is to give you a sketch of the proof of

Theorem. $d = 3$ *provided* $(a_\varepsilon)_\varepsilon$ *is choses approp. there exist a stationary in space and time solution to the limit SPDE and moreover the law of the solution at any given time in a non-Gaussian EQFT* ν *(without rotation invariance). We can prove it satisfies an IBP formula:*

$$\int \nabla_\varphi F(\varphi)\nu(\mathrm{d}\varphi) = \int F(\varphi)(-(m^2 - \Delta)\varphi - :\varphi^3:)\nu(\mathrm{d}\varphi).$$

[details in Gubinelli-Hofmanova CMP 2021, “A PDE construction…”]

▹ Ito & Dœblin introduced a variety of analysis adapted to the sample paths of a stochastic process.

▹ consider a family of kernels $(P_t)_{t \geqslant 0}$ on $\mathbb{R}^d$ satisfying Chapman–Kolmogorov equation

$$P_{t+s}(x, \mathrm{d}y) = \int P_s(x, \mathrm{d}z) P_t(z, \mathrm{d}y)$$

which defines a probability $\mathbb{P}$ on $C(\mathbb{R}_{\geqslant 0}, \mathbb{R}^d)$: the law of a continuous Markov process.

▹ sample paths have a "*tangent*" *process*. Ito identified it as a particular Lévy process: the Brownian motion $(W_t)_t$.

▹ stochastic calculus: from the local picture to the global structure via *stochastic differential equation* (SDE)

$$\mathrm{d}X_t = a(X_t)\mathrm{d}W_t + b(X_t)\mathrm{d}t$$

▹ these are the basic building blocks of **stochastic analysis**

▹ like in analysis, the fact that we can consider infinitesimal changes simplify the analysis and make appear universal underlying objects:

- polynomials → calculus, Taylor expansion
- Brownian motion and its functionals → Ito calculus, stochastic Taylor expansion

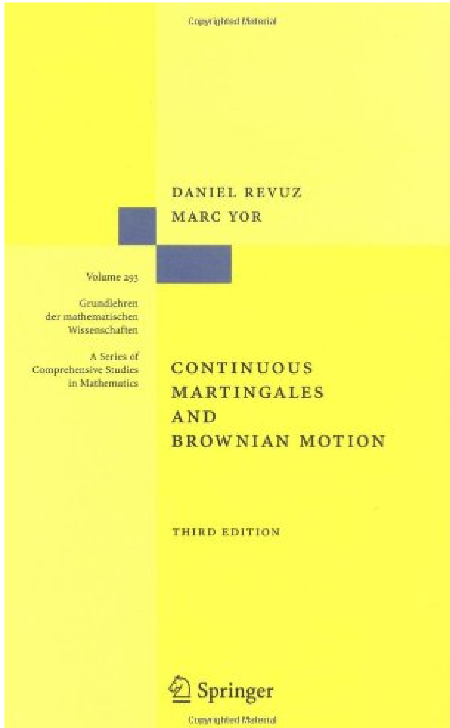
to have an analysis we need:

- a **change parameter** along which consider "change" (*time* for diffusions)
- a suitable **building block** for the infinitesimal changes (*Brownian motion* for diffusion)

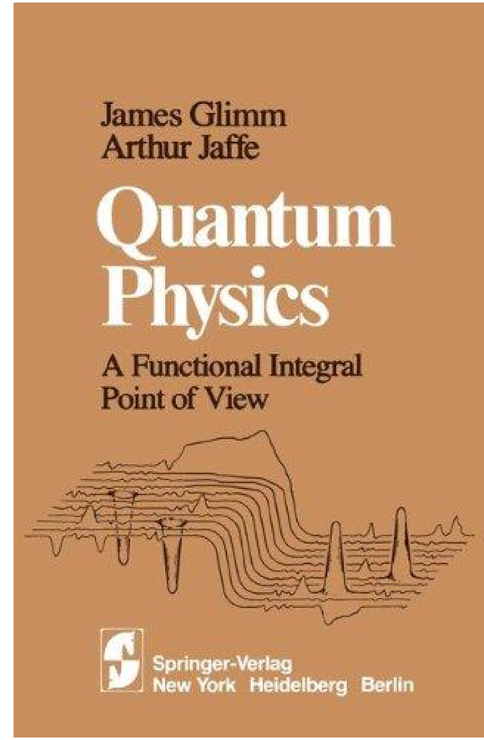
▹ other examples: rough paths, regularity structures, SLE,...

Newton's calculus		Ito's calculus
planet orbit	object	Markov diffusion
$(x,y) \in \mathcal{O} \subseteq \mathbb{R}^2$	global description	$P_t(x, \mathrm{d}y)$
$\alpha(x-x_0)^2 + \beta(y-y_0)^2 = \gamma$	·	$P_{t+s}(x,\mathrm{d}y) = \int P_s(x,\mathrm{d}z) P_t(z,\mathrm{d}y)$
t	change parameter	t
$x(t+\delta t) \approx x(t) + a\,\delta t + o(\delta t)$	local description	$P_{\delta t}(x,\mathrm{d}y) \approx e^{-\frac{(y-x-b(x)\delta t) a(x)^{-1} (y-x-b(x)\delta t)}{2\delta t}} \frac{\mathrm{d}y}{Z_x(\delta t)^{d/2}}$
$a\,t + b\,t^2 + \cdots$	building block	$(W_t)_t$
$(\ddot{x}(t), \ddot{y}(t)) = F(x(t), y(t))$	local/global link	$\mathrm{d}X_t = a(X_t)\mathrm{d}W_t + b(X_t)\mathrm{d}t$

Ito's calculus		stoch. quantisation
Markov diffusion	object	EQF
$P_t(x,\mathrm{d}y)$	global description	$\nu \in \mathrm{Prob}(\mathscr{S}'(\mathbb{R}^d))$
$P_{t+s}(x,\mathrm{d}y)=\int P_s(x,\mathrm{d}z)P_t(z,\mathrm{d}y)$	·	$\frac{1}{Z}\int_{\mathscr{S}'(\mathbb{R}^d)} O(\varphi)e^{-S(\varphi)}\mathrm{d}\varphi$
t	change parameter	t
$P_{\delta t}(x,\mathrm{d}y)\approx e^{-\frac{(y-x-b(x)\delta t)a(x)^{-1}(y-x-b(x)\delta t)}{2\delta t}}\frac{\mathrm{d}y}{Z_x(\delta t)^{d/2}},$	local description	$\phi(t+\delta t)\approx \alpha\phi(t)+\beta\delta X(t)+\cdots$
$(W_t)_t$	building block	$(X(t))_t$ $\partial_t X=\frac{1}{2}[(\Delta_x-m^2)X]+2^{1/2}\xi$
$\mathrm{d}X_t=a(X_t)\mathrm{d}W_t+b(X_t)\mathrm{d}t$	local/global link	$\partial_t\phi=\frac{1}{2}[(\Delta_x-m^2)\phi-V'(\phi)]+2^{1/2}\xi$



600 pages



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- **parabolic stochastic quantisation.** the parameter is an additional "fictious" coordinate $t \in \mathbb{R}$, playing the röle of a simulation time. The EQF is viewed as the invariant measure of a Markov process (SDE). Building block is a space-time white noise. [Parisi/Wu, Nelson, Jona-Lasinio/Mitter, Albeverio/Röckner, Da Prato/Debbusche, Hairer, Catellier/Chouk, Mourrat/Weber, G./Hofmanova, Albeverio/Kusuoka, Chandra/Moinat/Weber, Shen, Garban, many others...]

$$\partial_t \phi = \frac{1}{2}[(\Delta_x - m^2)\phi - p'(\phi)] + 2^{1/2}\xi$$

- **canonical stochastic quantisation.** same as for parabolic, but the evolution takes place in "phase space" and the SDE is second order in time, giving rise to a stochastic wave equation. [G./Koch/Oh, Tolomeo, Oh/Robert/Wang]

$$\partial_t^2 \phi + \partial_t \phi = \frac{1}{2}[(\Delta_x - m^2)\phi - p'(\phi)] + 2^{1/2}\xi$$

- **elliptic stochastic quantisation.** the parameter is a coordinate $z \in \mathbb{R}^2$. Building block is a white noise in $\mathbb{R}^{d+2}$. An elliptic stochastic partial differential equation describes the EQF as a function of the white noise. Link with supersymmetry.

  [Parisi/Sourlas, Klein/Landau/Perez, Albeverio/De Vecchi/G., Barashkov/De Vecchi]

$$-\Delta_z \phi(z,x) = \frac{1}{2}[(\Delta_x - m^2)\phi(z,x) - p'(\phi(z,x))] + 2^{1/2}\xi(z,x)$$

- **variational method.** the parameter $t \geqslant 0$ is a energy scale. Building block is the Gaussian free field decomposed along t. The EQF is described as the solution of a stochastic optimal control problem. [Barashkov/G.]
- **rg method.** the parameter $t \geqslant 0$ is a energy scale. Building block is the Gaussian free field decomposed along t. The effective action of the EQF satisfies an Hamilton–Jacobi–Bellmann equation. [Wilson, Wegner, Polchinski, Salmhofer, Brydges/Kennedy, Mitter, Gawedzki/Kupiainen, Brydges/Bauerschmidt/Slade, Bauerschmidt/Bodineau, Bauerschmidt/Hofstetter, also many others...]

references

for more details on the history of EQFT and SQ look at the introductions of these papers:

- M. Gubinelli and M. Hofmanova, `A PDE Construction of the Euclidean Φ^4_3 Quantum Field Theory', *ArXiv:1810.01700 [Math-Ph]*, 3 October 2018, http://arxiv.org/abs/1810.01700.
- S. Albeverio, F. C. De Vecchi, and M. Gubinelli, `Elliptic Stochastic Quantization', *Annals of Probability* 48, no. 4 (July 2020): 1693–1741, https://doi.org/10.1214/19-AOP1404.
- S. Albeverio et al., `Grassmannian Stochastic Analysis and the Stochastic Quantization of Euclidean Fermions', *ArXiv:2004.09637 [Math-Ph]*, 25 May 2020, http://arxiv.org/abs/2004.09637.

for hyperbolic SQ and the variational method one could refer to

- M. Gubinelli, H. Koch, and T. Oh, `Renormalization of the Two-Dimensional Stochastic Nonlinear Wave Equations', *Transactions of the American Mathematical Society*, 2018, 1, https://doi.org/10.1090/tran/7452.
- N. Barashkov and M. Gubinelli, `A Variational Method for Φ^4_3 ', *Duke Mathematical Journal* 169, no. 17 (November 2020): 3339–3415, https://doi.org/10.1215/00127094-2020-0029.

…end of lecture 1