Branching random walks, continuation

Remark

If $\mathbb{P}[X \geq t] = e^{-ct^r}$

where $0 < r < 1$ and $E[X^2] < \infty$

then $\frac{1}{n} \log \mathbb{P}[S_n \geq n^{1/r} y] \xrightarrow[n \to \infty]{} -cy^r$

$\xRightarrow[\text{arguments}]{\text{same}}$ $\quad \frac{M_n}{n^{1/r}} \longrightarrow x^* = \left(\frac{\log m}{c}\right)^{1/r}$

"Stretched exponentials tails"

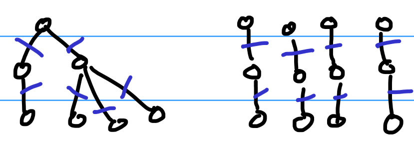
$\longrightarrow$ friday, Piotr Dyszewski.

Remark

$X, X_1, X_2, \dots$ iid, $\quad S_n = \sum_{i=1}^{n} X_i$

$S_n, S_n^{(1)}, S_n^{(2)}, \dots$ iid

$$\widetilde{M}_n = \max_{1 \leq i \leq |D_n|} S_n^{(i)}$$



Showed $\left(\mathbb{P}[\widetilde{M}_n \geq ny] \leq m^n \, \mathbb{P}[S_n \geq ny]\right)$

$$\limsup \frac{\widetilde{M}_n}{n} \leq x^*$$

Lemma 1 $\quad M_n \leq \widetilde{M}_n$

Lemma 2 $\quad (X_i)_{i \geq 1}, (Y_i)_{i \geq 1}$ indep.

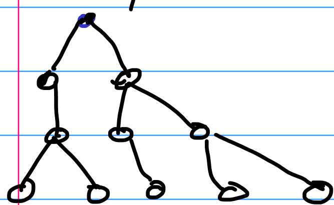
$(Y_i)_{i\ge 1}$ identically distributed.

Then

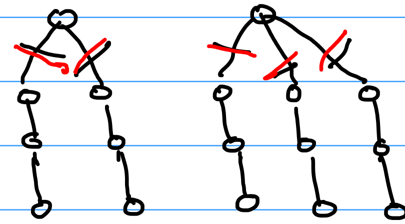
$$\max_{1\le i\le n}(X_i+Y_1) \le \max_{1\le i\le n}(X_i+Y_i)$$

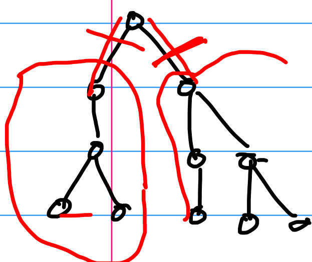
Exercise: Pf.

Proof of Lemma 1 by induction:

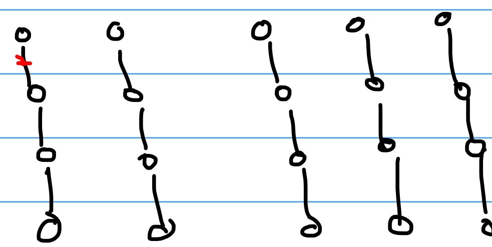


$\le$ Ind. hyp.





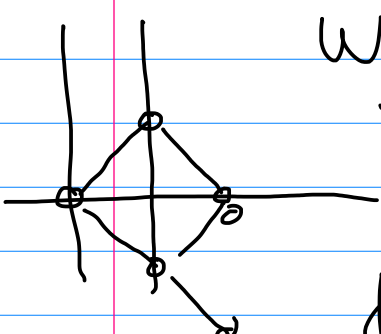
$\le$ Lemma 2



Exercises

(i) Show $\frac{\widetilde{M}_n}{n} \longrightarrow x^*$ $(0\le\alpha\le 1)$

(ii) $\mathbb{P}[X=1]=\alpha=1-\mathbb{P}[X=-1]$



Will the cloud visit 0 infinitely often, i.e. look at

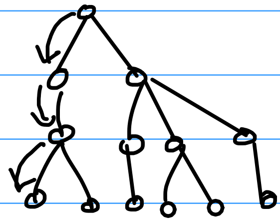
$\gamma=\mathbb{P}[S_v=0$ for some $v\in D_n$, for inf. many $n]$.

- Find conditions on α s.t. $\gamma=0$ or such that $\gamma>0$.

(conditions on

α and m)

(iii) Tree-indexed MC



particles

- reproduce according to $p(\cdot)$
- offspring takes a step according to $q(x,\cdot)$ if mother is at x

$q(x,y)_{x,y\in S}$ are trans. prob. of an irreduc. MC, S countable.

Take $0 \in S$, define γ as before

Find conditions on m, $q(\cdot,\cdot)$ s.t. $\gamma = 0$ or $\gamma > 0$.

Second order, i.e. $M_n - n x^* \sim$?

E. Aïdékon proved

Thm

X non-lattice

$\mathbb{E}[e^{\lambda X}] < \infty \quad \forall \lambda \in \mathbb{R}$

$\log m \in \operatorname{int}\{v : I(v) < \infty\}$

$\mathbb{E}[|\Delta_1|^{1+\delta}] < \infty$ for some $\delta > 0$

Then

$$\mathbb{P}\left[M_n - x^* n + \frac{3}{2\bar{\lambda}} \log n \leq t\right] \xrightarrow[n \to \infty]{} \mathbb{E}\left[\exp(-C Z_\infty e^{-t})\right]$$

where $\bar{\lambda} = I'(x^*)$

and Z_∞ is the a.s. limit of

$$Z_n = -\sum_{v \in D_n} (S_v - n x^*) e^{\bar{\lambda}(S_v - n x^*)}$$

Exercise

Check that (Z_n) is a martingale w.r.t. $(\mathcal{F}_n)$, $\mathcal{F}_n = \sigma(\text{"up to level } n\text{"})$

"derivative martingale"

Easier Theorem:

Thm Same assumptions, $p(m) = 1$. Then

$$\mathbb{P}\left[\tilde{M}_n - x^* n + \frac{1}{2\bar{\lambda}} \log n \leq t\right] \xrightarrow[n \to \infty]{} \exp(-C e^{-\bar{\lambda} t})$$

Sketch of proof

Note that for $a_n = o(\sqrt{n})$,

$$\mathbb{P}[S_n > nx^* - a_n]$$

$$\approx \frac{C}{\sqrt{n}} e^{-nI\left(x^* - \frac{a_n}{n}\right)} \quad \text{(Bahadur-Rao)}$$

non-lattice

But $nI\left(x^* - \frac{a_n}{n}\right) = n\underbrace{I(x^*)}_{\log m} - \underbrace{I'(x^*)}_{\bar{\lambda}} a_n + o(1)$

$$\Rightarrow \mathbb{P}[\tilde{M}_n \leq nx^* - a_n]$$

$$\sim \left(1 - \frac{C}{m^n \sqrt{n}} e^{\bar{\lambda} a_n + o(1)}\right)^{m^n}$$

Choose $a_n = \frac{\log n}{2\bar{\lambda}} - t$

$$\Rightarrow \mathbb{P}[\tilde{M}_n \leq nx^* - a_n]$$

$$\sim \exp\left(-Ce^{-\bar{\lambda} t} + o(1)\right)$$

$\blacksquare$

Exercise

(i) Make this precise for $X \overset{d}{=} N(0,1)$.

(ii) Prove that $E\left[\frac{\tilde{M}_n}{n}\right] \longrightarrow x^*$ and conclude

$$\limsup_{n\to\infty} \mathbb{E}\left[\frac{M_n}{n}\right] \le x^* .$$

Open questions

plenty for $d \ge 2$!

Field starts with BBM:

J. Berestycki, B. Mallein,

J. Schweinsberg, ...