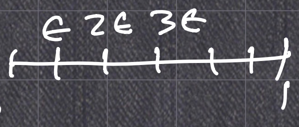
ρ_{ij} up to an ε error)



- If the union bd is over a not too big set, the upper bd one gets is

$$e^{-\phi(H,n,p,\delta) + \text{smaller order}}$$

- This fails if p is going to zero with n faster than a polylog.
- Full LDP on graphons for a fixed p was proven by Chatterjee & Varadhan (2011)
- The argument above which is more combinatorial. - Lubetzky-Zhao (2015)).

Lecture 2 07/16

- For p not too small

$$P\big(T(H,G) \geq (1+\delta)\, E(T(H,G))\big)$$
$$\approx e^{-\phi(H,n,p,\delta) + \text{corr}}$$

$$\phi(H,n,p,\delta) = \inf_{\substack{Q \\ \text{weighted} \\ \text{graph}}} \left(I_p(Q_n) : t(H,Q) \geq (1+\delta) E(T(G_{n,p})) \right)$$

$$\underline{I_p(Q_n)} = \frac{1}{2} \sum_{i \neq j} I_p(q_{ij})$$

What is $\phi(H,n,p,\delta)$?

- p fixed.

There is a graphon formulation

$$\phi(H,p,\delta) = \inf_{Q} \left(I_p(Q) : \downarrow \right)$$

$$\frac{1}{2} \iint I_p(Q(x,y))$$

When is $\phi(H,p,\delta)$ attained at a constant function.

- same as roughly saying, the large deviation event makes $G(n,p)$ behave like $G(n,r)$ for some $r > p$.

- Lubetzky-Zhao (2012) using a generalized form of Hölder's ineq found the above region exactly when H is regular.

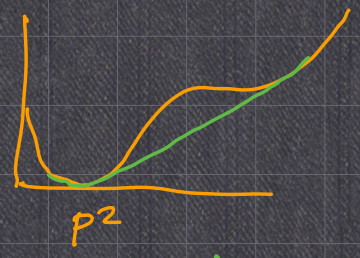
$H = K_3$ typical density is p^3.

want to achieve density r^3 $(r > p)$.

$x \to I_p(x^{1/d})$ – d-degree.

$$x \to I_p(\sqrt{x})$$

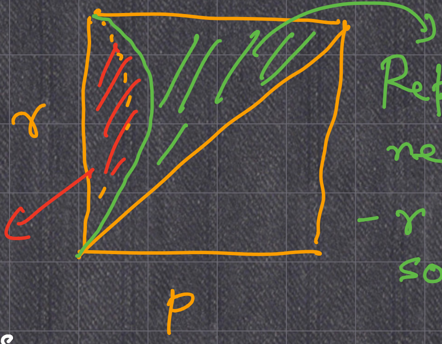
If $(r^2, I_p(r))$ lies on the convex minorant.



Consider Convex Minorant

Then the constat fuction r is the optimal solution.

– violated, then there is a block graphon construction which does better.



There are better Candidates.

Replica-sym region – r is the solution.

– Hölder's ineq allows to pass from homomorphism densities to various norms of the graphon.

– Open problem – To find the exact phase diag for any conn. non-regular graph.

– (with B. Bhattacharya, we have an unpublished result which improves the naive

Hölder bound for K_{12})

- simulations predict that optimizers in the sym breaking region should be stochastic block models with 2 blocks.

$$\begin{array}{|c|c|} \hline p_1 & p_2 \\ \hline p_2 & p_3 \\ \hline \end{array}$$

Recall a standard way to prove large deviation bounds is by computing exp moments.

- This naturally leads to study of a class of Gibbs meas. on graphs

$$\pi(G) \sim e^{n^2 H(G)}$$

Z - normalizing const

$\psi = \log(Z)$

$$H(G) = \sum_{i=1}^{s} \beta_i \, t(H_i, G)$$

Fix graphs $H_1, H_2, \ldots H_s$.

If $\beta_i > 0$ large. Then more mass on G with high sub-graph densities. - (Exp. Random graph model (ERGM).

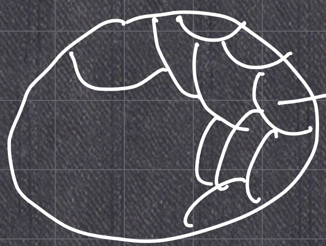
If $\beta_i < 0$, then subgraphs are avoided.

- Large deviation theory can be used to study ERGM, in particular ψ.

$$Z = \frac{1}{2^{n^2}} \sum_{G} e^{n^2 \sum B_i\, t(H_i, G)}$$

$$\psi \approx n^2 \left[\sup_{w} H(w) - I_{1/2}(w) \right]$$

– Chatt–Diaconis (2013)



$\rightarrow$ H does not oss. much.

- If $B_i > 0$, the optimizers are const.

- ERGM looks like a mix of Erdos–Renyi graphs

— (Eldan, Eldan Gross 2018-19)

- (Bhamidi – Breslen Sky) which looked at high & low temp ERGM.

unique sol
Fast mixing of Glauber-dyn

mul-solutions
- slow mix.

$O(n^2 \log n)$ $\geq e^{\Omega(n)}$

P_α ERGM $\overset{?}{\sim}$ $G(n, p_*)$

close in
cut distance.

- Conc of measure
- Central limit theorem
  for no of edges.
- (with K. Nam 2019)

Gauss. conc of measure for Lipschitz function

- Stein's method of
  ex. pair.

- If you take edges

  $e_1, e_2, \ldots e_M$ $M = o(n)$
  - vertex disj

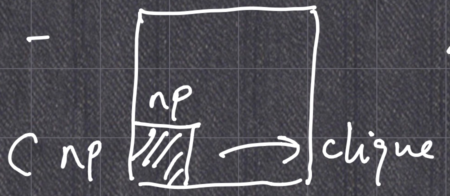
  $\Sigma 1_{e_i}$ satisfies a CLT

- Open prob. (Prove a full CLT).
  for the total no of edges.

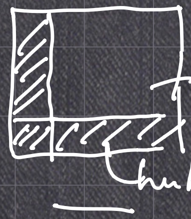
— Let's go back to $\phi(H, n, p, \delta)$
we will consider the case when
$p \to 0$ with n.

- Const fuction is never the solution

- cost of making the total no of edges $\frac{n^2}{2}(1+\delta)p \approx e^{-n^2 p C_\delta}$

- 

$\approx n^2p^2$ no of edges introduced but in a very compact fashion.



Cost $\approx p^{n^2p^2}$
$= e^{-n^2p^2 \log(1/p)}$

$n^2 p \gg n^2 p^2 \log\left(\frac{1}{p}\right)$

$\phi(H, n, p, \delta)$

H is conn. of max degree Δ.
H^* is the induced sub-graph on degree Δ vertices.

$I_{H^*}(x) =$ independence poly of H^*
$= \sum_k i_{H^*}(k)\, x^k.$

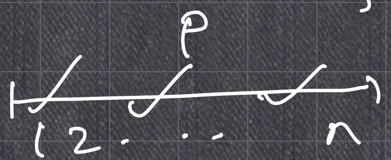
$i_{H^*}(k) = \#$ of ind sets of H^* of size k.

(ind set is a set of ver. with no edges between them.)

$p \gg n^{-1/\Delta}$ $\quad I_{H^*}(\theta) = 1+\delta$

$$\frac{\Phi(H, n, p, \delta)}{n^2 p^{\Delta} \log(1/p)} = \begin{cases} \min\left(\theta, \frac{1}{2}\delta^{2/v(H)}\right) \\ \theta \quad H \text{ is irr.} \end{cases}$$

- (Bhattachary, G., Lubetzky-Zhao 2016)
  - Cliques (L.Z 2015)
- Arithmetic progressions in random subsets of $[1, \dots n]$ or $\mathbb{Z}/n$



$S \subset \{1, \dots n\}$

T_k = # of AP of length k in S.

- (Bhattachary, G. Shao, Zhao 2017)
  - we proved precise asymp. for the corr. LD problem.
- Related to the following
  - given m. which subset $A \subset \mathbb{Z}$ of size m maximizes no of k AP's in A.

the interval is extremal. $\leftarrow$ $-K=3$ (Green-Sisask)

we extended to all k.

- If H is of max deg. Δ

$$p \gg \frac{1}{n^{1/\Delta}}$$

If H is Δ-regular.

$p \gg \frac{\text{polylog}}{n^{2/\Delta}}$ (Basak – Basu)

following (Harel – Mousset – Samotij) who settled the prob for cliques

beyond this random variables start looking Poisson & that govern LDP.